Supersymmetry

1 Motivations.

Out of a number of possible motivations to study SUSY we'll look (very briefly) at one - a hierarchy problem. More precisely, in a SM context, this problem is referred to as a "gauge hierarchy problem". This is a problem of a failure of dim analysis: scale EWSB is much-much smaller than Planck scale.

We'll look at a toy model of a single complex scalar φ interacting with a Dirac fermion Ψ. The $\mathcal{L}$ will be chosen to be:

$$\mathcal{L} = |\partial_\mu \varphi|^2 + i\bar{\Psi}\not{\partial}\Psi - \lambda^2|\varphi|^4 - y\varphi\bar{\Psi}\frac{1-\gamma_5}{2}\Psi - y\varphi^*\bar{\Psi}\frac{1+\gamma^5}{2}\Psi$$

We can ask about renormalization, specifically abot renormalized scalar mass^2

At one loop:

Fermionic & Bosonic contributions are given by

$$-i\delta m^2|_{ferm} = (-1)$$

$$= -2|y|^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2} \sim i\,\frac{2|y|^2}{16\pi^2}\Lambda^2$$

$$-i\delta m^2|_{scalar} = 4\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}(-i\lambda^2) = -i\,\frac{4\lambda^2}{16\pi^2}\Lambda^2$$

The mass correction @ 1 loop:

$$\delta m^2 \sim \frac{1}{16\pi^2}(4\lambda^2 - 2y^2)\Lambda^2$$

quadratically divergent unless $y = \sqrt{2}\lambda$. Even then, 2-loop correction is quadratically divergent ~~tune~~ $O((\frac{1}{16\pi^2})^2)$. Can tune y & λ again and again. This is because QFT generally generates all terms in a $\mathcal{L}$ that are allowed by symmetry.

Even if we tuned λ & y to keep φ massless/light, Adding heavy fields to the model with masses $O(\Lambda)$ that interact with φ, Ψ will change RG evolution of λ, y so that m_φ^2 is large again. So in this model low energy physics is extremely sensitive to UV physics @ arbitrarily high energy scales.

Shirman, Lectures

~~We would like~~

This is in contrast with the mass of our fermion: due to presence of chiral symmetry, fermion will be massless. If $\langle\varphi\rangle\neq 0$ chiral symmetry will be spontaneously broken but

$$\delta m_\psi \sim \langle\varphi\rangle \ln\frac{\langle\varphi\rangle}{\Lambda}$$

→ can be small as long $\langle\varphi\rangle$ is small

But $\langle\varphi\rangle$ (if non-zero) tends to be large.

In non-Abelian gauge theories, chiral symmetry may be broken ~~spontaneously~~ dynamically $\sim \Lambda e^{-\frac{8\pi^2}{g^2}}$ ~~at~~ and exponentially low symmetry breaking scale is natural. Light fermions are then naturally light.

Would like something like this as an extension of MSSM. New physics @ $\Lambda_{new} \sim$ TeV explains EWSB. Scale of new physics Λ_{new} has to be stable (no repeat of a SM hierarchy problem) Ideally, scale of new physics is generated dynamically to explain why it is small.

SUSY fits the bill.

2. Weyl fermions

We begin with technicalities. Consider a free Dirac fermion:

$$\mathcal{L} = i\bar\Psi\partial_\mu\gamma^\mu\Psi - m\bar\Psi\Psi$$

We often write it in terms of left-handed and right handed components

$$\Psi = \begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix}$$

Instead we'll use $\Psi = \begin{pmatrix}\eta^\alpha\\ \bar\chi^{*}_{\dot\alpha}\end{pmatrix}$, $\alpha,\dot\alpha = 1,2$

Where η, χ are left-handed while η^*, χ^* are right-handed. Spinorial indices on lh fermions will always be undotted while indeces on rh fermions dotted (a bit superfluos notation)

In the Weyl basis

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}, \quad \sigma^0 = I_2 \quad \sigma^i \text{ - Pauli matric}$$

$$\sigma^\mu = (1, \sigma^i) \quad \bar\sigma^\mu = (1, -\sigma^i)$$

Lagrangian can nicely be written in terms of Weyl fermions as

$$\mathcal{L} = i\eta^*\partial_\mu\bar\sigma^\mu\eta + i\bar\chi^*\partial_\mu\bar\sigma^\mu\bar\chi - m\bar\chi\eta - m\bar\chi^*\eta_*$$

where we integrated by parts to write kinetic terms in a similar ways.

Notes to be mentioned earlier:
If Ψ transforms in a definite way under some global or local symmetry, η & χ^* have the same charge. This means η & χ have opposite charges.

We'll use $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$ to lower and raise indices.

$$\eta^\alpha = \varepsilon^{\alpha\beta}\eta_\alpha$$

Add summations

$$\left.\begin{array}{l} \eta^\alpha \eta_\alpha = \varepsilon^{\alpha\beta}\eta_\alpha\eta_\beta \leftarrow \\ \eta^\alpha \bar\chi_\beta = \varepsilon^{\alpha\beta}\eta^\alpha \bar\chi_\beta = \bar\chi^\alpha \eta_\alpha \leftarrow \end{array}\right\} \text{Lorentz invariants}$$

Lorentz vectors:

$$\eta^*_\alpha \bar\sigma^{\mu\dot\alpha\alpha}\eta_\alpha \quad \text{or} \quad \eta^\alpha \sigma^\mu_{\alpha\dot\alpha}\eta^{*\dot\alpha}$$

Some useful relations to be found in e.g. Wess & Bagger

$$\{\sigma^\mu, \sigma^\nu\}_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta$$

Propagators: (for a fermion with dirac mass)

~~$\langle \psi\psi^*$~~

$$\langle \eta_\alpha \eta^*_{\dot\beta} \rangle = \frac{i p_\mu \sigma^\mu_{\alpha\dot\beta}}{p^2 - m^2}$$

$$\langle \chi_\alpha \eta^\beta \rangle = \frac{i m \delta_\alpha{}^\beta}{p^2 - m^2}$$

$$\langle \chi^{*\dot\alpha}_\alpha \eta^*_{\dot\beta} \rangle = \frac{i m \delta^{\dot\alpha}{}_{\dot\beta}}{p^2 - m^2}$$

In a massless free theory η & χ decouple completely. Thus a minimal spinor in 4d is not a Dirac spinor but Weyl spinor.

3 SUSY algebra.

We want ~~sy~~ to construct supersymmetry - sym. between bosons & fermions. This means that generators of the symmetry must act as

$$Q\,|\text{boson}\rangle \sim |\text{fermion}\rangle$$

$$Q\,|\text{fermion}\rangle \sim |\text{boson}\rangle$$

Lorentz transformation properties must be the same on both sides. So Q is a spinor.

Since minimal spinor in 4d is a Weyl spinor, Q must be at least a single Weyl spinor. Since Weyl spinor has 4 real components, minimal SUSY has 4 supercharges.

If symmetry is a true symmetry of the Lagrangian one should be able to define an algebra.

Indeed, such an algebra exists:

$$\{Q_\alpha Q^*_{\dot\alpha}\} = 2\hat p^\mu \sigma^\mu_{\mu\alpha\dot\alpha}$$

$$\{Q_\alpha Q_\beta\} = \{Q^*_{\dot\alpha} Q^*_{\dot\beta}\} = [Q_\alpha \hat P^\mu]$$

We will define SUSY generators ~~more con~~ explicitly later

Since SUSY algebra involve generators of the Poincare group and since coordinate transformations are a local symmetry, supersymmetry also must be a local symmetry. ~~So~~ In particular, even graviton must have a superpartner - gravitino.

Graded Lie algebra that we have written down is a unique (under rather general assumptions) extension of a Poincare symmetry. The only further generalization that is allowed is increasing number of symmetry generators. 4 generators ($N=1$ SUSY) require existence of particles related by symmetry with spins different by 1/2, eg.
scalar and fermion; fermion & vector
spin 3/2 fermion & gravitino.

With 8 generators ($N=2$) one could have a multiplet with spin 0, 2×spin 1/2, vector
OR 2×spin 0, 2×spin 1/2.
With 16 generators ($N=4$) all multiplets must have at least spin 1 particles.

With

SUSY breaking first look.

Ground state of a theory spontaneously breaks ~~St~~ a symmetry iff ground state is not annihilated by the symmetry generator:

$$Q_\alpha |0\rangle \neq 0 \quad Q_{\dot\alpha}^\dagger |0\rangle = 0$$ is condition of an unbroken SUSY

Lookin at the algebra:

$$\sum_{\alpha=\dot\alpha} Q_\alpha Q_{\dot\alpha}^\dagger + Q_{\dot\alpha}^\dagger Q_\alpha = 4P_0$$

Take vev:

$$\langle 0| \sum_{\alpha=\dot\alpha} Q_\alpha Q_{\dot\alpha}^\dagger + Q_{\dot\alpha}^\dagger Q_\alpha |0\rangle = 4\langle 0| P_0 |0\rangle = 4E$$

SUSY is spontaneously broken iff vacuum energy is non-zero.

Subtlety when M_{pl} is finite.

SUSY Lagrangians:

Start with the simplest smallest non-interacting theory containing 1 fermion:

	on shell	off-shell
ψ	2	4
φ	2	2
F	0	2

Need an auxiliary scalar to match off-shell dofs. No on-shell dof → no kinetic terms, algebraic equations of motion.

$$\mathcal{L} = |\partial_\mu \varphi|^2 + i\psi^\dagger \partial_\mu \bar{\sigma} \psi + F^\dagger F$$

Eoms: $F = 0$

SUSY variations

$$\delta\varphi = \epsilon^\alpha \psi_\alpha \qquad \delta\varphi^\dagger = \epsilon^\dagger_{\dot\alpha} \psi^{\dagger\dot\alpha}$$

$$\delta\psi_\alpha = -i\sigma^\nu_{\alpha\dot\alpha} \epsilon^{\dagger\dot\alpha} \partial_\nu \varphi + \epsilon_\alpha F \qquad \delta\psi^\dagger_{\dot\alpha} = i\epsilon^\alpha \sigma^\nu_{\alpha\dot\alpha} \partial_\nu \varphi^\dagger + \epsilon^\dagger_{\dot\alpha} F^\dagger$$

$$\delta F = -i\epsilon^\dagger_{\dot\alpha} \bar\sigma^{\dot\alpha\alpha} \partial_\mu \psi_\alpha \qquad \delta F^\dagger = i\partial_\mu \psi^\dagger_{\dot\alpha} \epsilon^{\dagger\dot\alpha}$$

Can easily check that $\delta\mathcal{L}$ is a total derivative. SUSY is a symmetry.

Another observation δF is a total derivative. Adding

$$\Delta\mathcal{L} = \mu^2 F + \mu^{*2} F^*$$ keeps $\mathcal{L}$ supersymmetric.

Let's solve eom's now

$$F = -|\mu|^2 \qquad V = |\mu|^4$$

Non-vanishing ground state energy. SUSY must be broken.

Another way to see this: when $F \neq 0$ variations of fields in the ground state are non-vanishing.

But this is still a free theory! In the absence of gravity zero point energy is unobservable! So is SUSY broken? Gravity is necessary in SUSY, theory is interacting due to gravity, order parameter for SUSY is F/M_{pl}^2

Interacting theory

Using SUSY transformations we can go ahead & start constructing SUSY Lagrangians term by term:

E.g. Add interaction ~~$\varphi\psi\psi$~~ $\varphi\psi\psi$

$$\delta(\varphi\psi\psi) \supset \varphi\psi^\alpha\partial_\mu\varphi\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} + \varphi(\psi\epsilon)F$$

Term $\delta(F\varphi^2) \supset \varphi^2(\bar\epsilon\bar\sigma^\mu\partial_\mu\psi) + F\varphi(\epsilon\psi)$

Sum gives a total derivative.

It is better to find a more efficient method of writing SUSY Lagrangians. For that purpose, combine fields related by SUSY into SUSY multiplets (or supermultiplets for short).

E.g. (φ, ψ, F)

It is convenient to write supermultiplet as a single superfield. For that purpose, introduce anticommuting coordinates

$$\{x_\mu\} \to \{x_\mu, \theta_\alpha, \bar\theta_{\dot\alpha}\}$$

$$\{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \bar\theta_{\dot\beta}\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0$$

Integrals $\int d\theta^\alpha = 0 \quad \int d\theta^\alpha\theta_\beta = \delta^\alpha_\beta$

Define $\int d^2\theta = -\frac{1}{4}\int\epsilon_{\alpha\beta}\,d\theta^\alpha d\theta^\beta$.

Then

$$\int d^2\theta\,\theta^2 = -\frac{1}{4}\epsilon_{\alpha\beta}\epsilon^{\sigma\tau}\int d\theta^\alpha d\theta^\beta\theta_\sigma\theta_\tau =$$

$$= -\frac{1}{4}\left(\epsilon_{\alpha\beta}\epsilon^{\sigma\tau}\delta^\beta_\sigma\delta^\alpha_\tau - \epsilon_{\alpha\beta}\epsilon^{\sigma\tau}\delta^\beta_\tau\delta^\alpha_\sigma\right) = +1$$

$$\int d^2\theta(\psi\theta)(\chi\theta) = \epsilon^{\alpha\beta}\epsilon^{\sigma\tau}\psi_\alpha\chi_\sigma\int\left(-\frac{1}{4}\right)\epsilon_{\mu\nu}\,d\theta^\mu d\theta^\nu\theta_\beta\theta_\tau =$$

$$\left(-\frac{1}{4}\right)\epsilon^{\alpha\beta}\epsilon^{\sigma\tau}\epsilon_{\mu\nu}\left(\delta^\nu_\beta\delta^\mu_\tau - \delta^\mu_\beta\delta^\nu_\tau\right)\psi_\alpha\chi_\sigma$$

$$= -\frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\sigma\tau}\epsilon_{\tau\beta}\psi_\alpha\chi_\sigma = -\frac{1}{2}\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\frac{1}{2}(\psi\chi)$$

We are now tempted to write

$$\Phi = \varphi + \sqrt{2}\,\theta\psi + \theta^2 F$$

But this is not the most general superfield.

$$\Phi = \varphi + \sqrt{2}\,\theta\psi + \sqrt{2}\,\bar\theta\bar\chi + \bar\theta\bar\sigma^\mu\theta A_\mu + \theta^2 F + \bar\theta^2 G + \ldots$$

Need to restrict ourselves to irreducible representation

Introduce superspace derivatives.

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \quad \bar D_{\dot\alpha} = -\partial_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \quad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0$$

Aside: SUSY charges:

$$Q_\alpha = \frac{\partial}{\partial\theta_\alpha} - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \qquad \bar Q_{\dot\alpha} = -\partial_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

Define a chiral superfield by a condition:

$$\bar D_{\dot\alpha}\Phi = 0 \leftarrow \text{chiral}$$

$$D_\alpha \Phi^{+} = 0 \leftarrow \Phi^{+} \text{ is anti-chiral}$$

How to solve: $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$

$$\bar D_{\dot\alpha} y^\mu = \left(-\frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\right)(x^\mu + i\theta\sigma^\mu\bar\theta) =$$

$$= i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha} = 0$$

As a result:

$$\Phi \equiv \Phi(y) = \varphi(y) + \sqrt{2}\,\theta\psi(y) + \theta^2 F(y)$$

$$\bar D_{\dot\alpha}\Phi(y) = \partial_{y^\mu}\Phi(y)\,\bar D_{\dot\alpha} y^\mu = 0$$

Also an analitic function of chiral superfields is in itself a chiral superfield

$$\bar D_{\dot\alpha} f(\Phi(y)) = 0$$

$$\Phi(x) = \varphi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\varphi + \frac{1}{4}\theta^2\bar\theta^2\partial^2\varphi +$$

$$+ \sqrt{2}\,\theta\psi - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar\theta + \theta^2 F$$

F term transforms into a total derivative, so θ^2 component of ~~an~~ chiral superfield is supersymmetric. More generally, θ^2 component of an analitic function of chiral superfield is supersymmetric

E.g. $\Phi_1\Phi_2|_{\theta^2} = F_1\varphi_2 + F_2\varphi_1 - \psi_1\psi_2$

$$\delta F_{\Phi_1\Phi_2} = \delta(F_1\varphi_2 + F_2\varphi_1 - \psi_1\psi_2) =$$

$$-i\varepsilon^{*}\bar\sigma^\mu(\partial_\mu\psi_1)\varphi_2 - i\varepsilon^{*}\bar\sigma^\mu(\partial_\mu\psi_2)\varphi_1 + F_1\varepsilon\psi_2 + F_2\varepsilon\psi_1 -$$

$$-\psi_1(-i\sigma^\mu\varepsilon^{*})\partial_\mu\varphi_2 - \psi_2(-i\sigma^\mu\varepsilon^{*})\partial_\mu\varphi_1 - (\psi_1\varepsilon)F_2 - (\psi_2\varepsilon)F_1$$

$$= \partial_\mu\left[-i(\varepsilon^{*}\sigma^\mu\psi_1)\varphi_2 - i\varepsilon^{*}\sigma^\mu\psi_2\varphi_1\right]$$

$$\Phi_1\Phi_2|_\theta = \sqrt{2}(\psi_1\varphi_2 + \psi_2\varphi_1)$$

$$\delta F_{f(\Phi)} = -i\varepsilon^{*}\bar\sigma^{\mu\dot\alpha}\partial_\mu\frac{1}{\sqrt{2}}\psi_{f(\Phi)}$$

SUSY terms in the Lagrangian:

$$\Delta \mathcal{L} = W(\Phi)|_{\theta^2} + h.c. = \int d^2\theta\, W(\Phi) + h.c.$$

using the fact that integration & differentiation give the same results.

$W(\Phi_i)$ is a superpotential.

E.g. $W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$

$$\Delta\mathcal{L} = mF\varphi + \lambda F\varphi^2 - \lambda\varphi\psi\psi - \frac{m}{2}\psi\psi$$

No kinetic terms yet.

We can check that kinetic terms of a free theory arise from the θ^4 component of

$$\Phi^+\Phi|_{\theta^4}$$

We already know that this transforms as a total derivative. More generally

$$\Phi_1^+\Phi_2(y) = (\varphi_1^* + i\theta\sigma^\mu\bar\theta\partial_\mu\varphi_1^* - \frac{1}{4}\bar\theta^2\theta^2\partial^2\varphi_1^* + \sqrt{2}\bar\theta\psi_1^* -$$
$$\frac{i}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\psi_1^*\bar\theta^2 + \bar\theta^2 F_1^*) \times$$
$$(\varphi_2 - i\theta\sigma^\mu\bar\theta\partial_\mu\varphi_2 - \frac{1}{4}\bar\theta^2\theta^2\partial^2\varphi_2 + \sqrt{2}\theta\psi_2 +$$
$$+\frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi_2\sigma^\mu\bar\theta + \theta^2 F_2)$$

θ^4 component of a real function of chiral superfields transforms into a total derivative

$$\Delta\mathcal{L} = \int d^4\theta\, K(\Phi^+, \Phi)$$

In particular, we know that

$$\int d^4\theta\, K_{ij}(\Phi^+, \varphi)\,\Phi^+\Phi \text{ contains kinetic terms}$$

$$= K_{\phi^+\phi}(\varphi^+, \varphi)\left[|\partial_\mu\varphi|^2 + i\psi^*\partial_\mu\bar\sigma^\mu\psi + F^*F\right]$$

$K_{\phi^+\phi}$ is a wave-function factor.

~~Wess-Zumi~~

$K_{\phi^+\phi} \neq 1$ means that kinetic terms are not canonically normalized. To obtain physical masses and couplings one needs to rescale fields to get canonically normalized kinetic terms

Wess-Zumino model

$$K = Z_0\Phi^+\Phi \quad W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$

$$\mathcal{L} = \int d^4\theta\, K(\Phi^+\Phi) + \int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, \bar W(\Phi^+)$$

$$\mathcal{L} = Z_0(|\partial_\mu\varphi|^2 + i\psi^*\partial_\mu\bar\sigma^\mu\psi + F^*F) +$$
$$\left[(mF\varphi + \lambda F\varphi^3 + \frac{m}{2}\psi\psi - \lambda\varphi\psi^2) + h.c.\right]$$

Note that scalar terms: $\cancel{\frac{\partial W}{\partial\Phi}\varphi}$

$$\left.\frac{\partial W}{\partial \Phi}\right|_{\Phi=\varphi} F$$

and fermion terms

$$-\frac{1}{2}\left.\frac{\partial W}{\partial \Phi \partial \Phi}\right|_{\Phi=\varphi} \psi\psi$$

$\mathcal{L}$ is quadratic in F and forthermore, F equations of motion are algebraic.

$$\left.\begin{aligned} Z_0 F^* + \frac{\partial W}{\partial \varphi} &= 0 \\ Z_0 F + \frac{\partial \bar{W}}{\partial \varphi^+} &= 0 \end{aligned}\right\} = \begin{aligned} F^* &= -\frac{\partial W}{\partial \Phi}\frac{1}{K_{\varphi^+\varphi}} \\ F &= -\frac{\partial \bar{W}}{\partial \varphi^+} K_{\varphi^+\varphi}^{-1} \end{aligned}$$

Substituting bacu we obtain $\mathcal{L}$:

$$\mathcal{L} = Z_0 \left(|\partial_\mu \varphi|^2 + i\psi^* \partial_\mu \bar{\sigma}^\mu \psi\right) - V - \left(\frac{1}{2} W'' \psi\psi + h.c.\right)$$

$$V = \frac{\partial \bar{W}}{\partial \varphi^+} K''^{-1} \frac{\partial W}{\partial \varphi} = \frac{1}{Z_0} |m\Phi + \lambda \varphi^2|^2$$

Generalizing to many chiral superfields is easy

$$\mathcal{L} = \int d^4\theta\, K(\{\Phi_i^+, \Phi_i\}) + \int d^2\theta\, W(\{\Phi_i\}) + \int d^2\bar{\theta}\, \bar{W}(\{\Phi_i^+\})$$

Kinetic terms arise from

$$\sum_{ij} K_{ij}(\{\varphi_k^+, \varphi_k\})\, \varphi_i^+ \varphi_j \qquad K_{ij} = \frac{\partial K}{\partial \varphi_i^+ \partial \varphi_j}$$

Aftre integration over superspace coordinates:

$$\mathcal{L} = K_{ij}(\{\varphi_k^+, \varphi_k\})\left(\partial_\mu \varphi_i^* \partial_\mu \varphi_j + i\psi_i^* \partial_\mu \bar{\sigma}^\mu \psi_j + F_i^* F_j\right)$$
$$+ \left(\frac{\partial W}{\partial \varphi_i} F^i - \frac{1}{2} W^{ij} \psi_i \psi_j + h.c.\right)$$

Scalar potential:

$$V = \frac{\partial \bar{W}}{\partial \varphi_i^+} K_{ij}^{-1} \frac{\partial W}{\partial \varphi_j}$$

Unless Kahler potential is singular, still SUSY iff F-terms are non-vanishing

A note on supergravity.
SUSY algebra involves generators of the Poincare group
Once we insist on general coordinate invariance,
Poincare transformations become a local symmetry
Thus supersymmetry must be a local symmetry.
SUSY requires that graviton has a superpartner-
gravitino, spin 3/2 fermion.

In the presence of gravity scalar potential of a SUSY theory take the form

$$V = e^{K/M_{pl}^2}\left[\hat{D}_i W \, K_{ij}^{-1} \, \hat{D}_j \bar{W} - \frac{3|W|^2}{M_p^2}\right]$$

$$\hat{D}_i W = \frac{\partial W}{\partial \varphi_i} + \frac{\partial K}{\partial \varphi_i}\frac{W}{M_{pl}^2} \leftarrow \text{covariant SUGRA derivative.}$$

Again orde parameter for SUSY Breaking is an F-term, now $\hat{D}_i W = 0$

It is possible that $\left.\begin{array}{l}\hat{D}_i W = 0 \\ W \neq 0\end{array}\right\}$ $\begin{array}{l}\leftarrow \text{SUSY unbroken} \\ \leftarrow \text{negative cosmological constant}\end{array}$

Theories with unbroken SUSY must have vanishing or negative cosmological constant.

Gauge interactions

Need another irrep of SUSY to construct a Lagrangian containing massless vector field

Degrees of freedom

		on-shell	off-shell	
	A_μ	2	3	SUSY transforms later
gaugino →	λ	2	4	
→	D	0	1	

↗ Need an auxiliary real scalar field

$$V = C(x) + i\theta\chi - i\bar{\theta}\chi^\dagger + \frac{i}{2}\theta^2[M+iN] - \frac{i}{2}\bar{\theta}^2[M-iN]$$

$$-\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}[\lambda^\dagger + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi]$$

$$-i\bar{\theta}^2\theta[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\chi^\dagger] + \frac{1}{2}\theta^2\bar{\theta}^2[D(x) + \frac{1}{2}\Box C(x)]$$

We have a vector living in $\theta\sigma^\mu\bar{\theta}$ component.

Under gauge symmetry $A_\mu \to A_\mu - \partial_\mu\alpha$

(Work in a normalization where kinetic term is $\frac{1}{4g^2}F_{\mu\nu}^2$)

Chiral superfield Λ has such a component

$$\Lambda \supset i\theta\sigma^\mu\bar{\theta}\partial_\mu(\alpha+i\beta)/2$$

Under the transformation

$$V \to V + i\Lambda - i\Lambda^\dagger : \quad A_\mu \to A_\mu - \partial_\mu\alpha$$

But other components of V also change.

$$C \to C - \beta(x)$$

~~$h \to h + \sqrt{2}\,\chi$~~

$$\chi \to \chi + \sqrt{2}\,h$$

$$M + iN \to M + iN + iF$$

$$\cancel{\chi^\dagger}\ \lambda + \frac{i}{2}\sigma^\mu \partial_\mu \chi^\dagger \to \lambda + \frac{i}{2}\sigma^\mu \partial_\mu \chi^\dagger + \frac{i}{\sqrt{2}}\sigma^\mu \partial_\mu h^\dagger$$

$$D + \frac{1}{2}\Box C \to D + \frac{1}{2}\Box(C - \beta(x))$$

Choose β, h, F so that $C = M = N = \chi = 0$

This is gauge fixing. ~~Whi~~ Since we have not set $\alpha(x)$ ~~gauge f~~ ordinary gauge freedom is still available. But Wess-Zumino gauge makes SUSY less explicit

Now need to construct SUSY Lagrangian

Define a chiral spinorial superfield

~~$W_\alpha = \bar{D}^2_{\dot\alpha} D_\alpha V$ is a chiral superfield~~

$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$ is a chiral superfield.

$W^\alpha W_\alpha$ is chiral, θ^2 component is SUSY invariant

$$W^\alpha W_\alpha = +2i\lambda\sigma^\mu\partial_\mu\lambda^\dagger - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D^2 + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$\int d^4x\, \mathcal{L} = \int d^4x \left[\frac{1}{4g^2}\int d^2\theta\, W_\alpha W^\alpha + \int d^2\bar\theta\, \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\right]$$

$$= \int d^4x \left(\frac{1}{2g^2}D^2 - \frac{1}{4g^2}F_{\mu\nu}^2 + i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda\right)$$

Covariant derivatives:

$$\Phi \to e^{-i\Lambda q}\Phi \qquad \Phi^\dagger \to \Phi e^{i\Lambda^\dagger q}$$

$$\int d^4\theta\, \Phi_i^\dagger e^{qV}\Phi_i + \left(\int d^2\theta\, \frac{1}{4g^2}W^\alpha W_\alpha + h.c.\right) +$$

$$+ \left(\int d^2\theta\, W(\Phi_i) + h.c.\right)$$

~~most general~~ U(1) invariant Lagrangian

Generalization to SU(N) (non-abelian groups)

$$V \to V^a T^a$$

$$\Phi^\dagger e^{qV}\Phi \to \Phi^\dagger e^{V^a T^a}\Phi$$

$$\frac{1}{4g^2}\int d^2\theta\, W_\alpha W^\alpha \to \frac{1}{2}\int d^2\theta\, \mathrm{Tr}\, W_\alpha W^\alpha$$

One more subtlety in U(1): V is gauge invariant A_μ shifts by $\partial_\mu\alpha$ but all other components of V do not shift under ordinary gauge transformation

D-term potential

In $U(1)$:

$$\frac{1}{2g^2}D^2 + \sum q_i\, \varphi_i^\dagger D \varphi_i$$

Leading to

$$V = \frac{g^2}{2}\sum\left(|\varphi_i|^2 q_i\right)^2$$

In non-abelian theory:

$$\frac{1}{2g^2}(D^a)^2 + D^a\left(\sum_f \varphi_f^\dagger T^a \varphi\right)$$

$$V = \frac{g^2}{2}\sum_a\left|\sum_f \varphi_f^\dagger T^a \varphi\right|^2$$

For fund + $\overline{\text{fund}}$: $V = \frac{g^2}{2}\sum_a \left|\varphi^\dagger T^a \varphi - \bar\varphi^* T^a \bar\varphi^\dagger\right|^2$

When is potential vanishing?

$$D^a = T^{a\,m}_n \underbrace{\left(\sum_f \varphi^{\dagger f n}\varphi^f_m - \sum \bar\varphi^{f n}\varphi^f_m\right)}_{\sim \alpha I}$$

Then using flavor symmetries:

a) $F < N$

$$\langle\varphi\rangle = \langle\bar\varphi^*\rangle = \overset{\longrightarrow F}{\begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ 0 & \cdots & 0 \\ \vdots & & \vdots \end{pmatrix}} \Big\downarrow N$$

~~b) $F \geq N$~~

2) $F \geq N$:

$$\varphi = \left(\begin{array}{ccc|c} v_1 & & & \\ & \ddots & & 0 \\ & & v_N & \end{array}\right) \qquad \bar\varphi^* = \left(\begin{array}{ccc|c} v_1+v & & & \\ & \ddots & & 0 \\ & & v_N+v & \end{array}\right)$$

SUSY transformations.

$$\delta A_\mu^a = -\frac{1}{\sqrt{2}}\left[\varepsilon^\dagger \bar\sigma_\mu \lambda^a + \lambda^{\dagger a}\bar\sigma^\mu \varepsilon\right]$$

$$\delta\lambda^a_\alpha = -\frac{i}{2\sqrt{2}}\left(\sigma^\mu\bar\sigma^\nu\varepsilon\right)_\alpha F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\varepsilon_\alpha D^a$$

$$\delta\lambda^{\dagger a}_{\dot\alpha} = \frac{i}{2\sqrt{2}}\left(\varepsilon^\dagger\bar\sigma^\nu\sigma^\mu\right)_{\dot\alpha} F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\varepsilon^\dagger_{\dot\alpha} D^a$$

$$\delta D^a = -\frac{i}{\sqrt{2}}\left(\varepsilon^\dagger\bar\sigma^\mu D_\mu\lambda^a - D_\mu\lambda^{\dagger a}\bar\sigma^\mu\varepsilon\right)$$

Non-renormalization theorems:

$$\mathcal{L} = |\partial_\mu\varphi|^2 + i\psi^\dagger\partial_\mu\bar{\sigma}^\mu\psi - |\lambda|^2|\varphi|^4 - (y\varphi\psi\psi + h.c.)$$

If $y = \lambda$, theory is supersymmetric, otherwise it is not.

No SUSY: Z_φ, Z_ψ, δ_λ, δ_y, δ_{m_φ}, $\underset{\overset{\shortparallel}{0}}{\delta_{m_\psi}}$ ← due to chiral symmetry

Due to SUSY

$Z_\varphi = Z_\psi$, δ_λ & δ_y related, δ_{m_φ}, δ_{m_ψ} related

δ_{φ^3} related δ_y & δ_m

But SUSY has stronger constraints:

δ_λ $y = \lambda$ $m_\varphi = m_\psi$, and cubic

$\delta_y = 0$ $\delta_m = 0$

The only non-vanishing renormalization is that of the kinetic term.

→ Just look at SUSY Lagrangian:

$$\int d^2\theta\, W(\Phi) = \int d^2\theta\left(\frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3\right) \rightarrow \int d^2\theta\left(\frac{m+\delta m}{2}\Phi^2 + \frac{\lambda+\delta_\lambda}{3}\Phi^3\right)$$

NR theorems

Before proving NR theorems, one more sym.: R-symmetry

$$\mathcal{L} = \int d^4\theta\, K(\Phi^\dagger, \Phi) + \int d^2\theta\, W + \int d^2\bar{\theta}\, \bar{W}$$

A potential for another symmetry: $\theta \rightarrow e^{i\alpha}\theta$

$\int d\theta\,\theta = 1$ $\cancel{d\theta}$ $R_\theta = 1$ $R_{d\theta} = -1$

If $W \rightarrow e^{2i\alpha}W$ then ~~actio~~ $\mathcal{L}$ is invariant.

$R_W = 2$

E.g. $W = \frac{\lambda}{3}\Phi^3$ $R_\Phi = 2/3$ $R_W = 2$.

or $W = \frac{m}{2}\Phi^2$ $R_\Phi = 1$ $R_W = 2$

However $W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$ → no R symmetry

$R_\Phi = R_\varphi = R_\psi + 1 = R_F + 2$

To prove NR theorem, use all symmetries, including R-symmetry.

Also promote $\mathcal{L}$ parameters to background superfields:

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$

Think of m, λ as vevs of heavy (therefore non-dynamical superfields)

Symmetries

	$U(1)$	$U(1)_R$
φ	1	1
m	-2	0
λ	-3	-1
$\theta\ W$	0	2

After renormalization:

$$K = K(\varphi^\dagger, \varphi, m^\dagger, m, \lambda^\dagger, \lambda)$$

$$W = W(\varphi, m, \lambda) \leftarrow \text{invariant under all the symmetries.}$$

$$W = \frac{m}{2}\varphi^2 f\left(\frac{\lambda\varphi}{m}\right)$$

$\lambda \to 0$ limit $\quad W = \frac{m}{2}\varphi^2 + \frac{\lambda}{3}\varphi^3 + O(\lambda^2)$

$$f = 1 + \frac{2}{3}\frac{\lambda\varphi}{m} + O(\lambda^2)$$

Furthermore: $m \to 0$ limit must be regular, no negative power of m:

$$W = \frac{m}{2}\varphi^2 + \frac{\lambda}{3}\varphi^3 \text{ is exact}$$

No higher order terms generated

But also no renormalizaton of m, λ.

Indeed if there was, e.g., renorm of m

$\delta m \sim \lambda^n$ but no such term exist.

On the other hand, K is real, can be a function of $|m|^2, |\lambda|^2$ which are invariant under all the symmetries. So

$$K = \varphi^\dagger\varphi \to Z(\mu, |m|, |\lambda|)\,\varphi^\dagger\varphi$$

Gauge coupling renormalization:

$$\frac{1}{4g^2}\int d^2\theta\, W^\alpha W_\alpha \quad ,$$

Define $\tau = \frac{8\pi^2}{g^2} + i\theta_{YM}$

$$\frac{1}{4g^2}\int d^2\theta\, W^\alpha W_\alpha \to \frac{1}{32\pi^2}\int d^2\theta\, \tau\, W^\alpha_\alpha W_\alpha$$

$$\supset \frac{1}{4g^2}F_{\mu\nu}^2 + \frac{\theta_{YM}}{32\pi^2}F\tilde{F}$$

$F\tilde{F}$ is a total derivative.

In abelian theory $\theta_{YM} \to \theta_{YM} + \text{const}$ is a symmetry

In non-abelian theory

$$\frac{\theta_{YM}}{32\pi^2}\int d^4x\, F\tilde{F} = n\theta_{YM}$$

n is a winding number & one needs to sum over all n. ~~∃ no~~ ~~the~~ Physics depends on θ_{YM} but periodic with period 2π

$$\int d^2\theta\, \tau\, W^\alpha W^\alpha \longrightarrow \int d^2\theta\,(\tau + \text{const})\, W^\alpha W_\alpha$$

$$\frac{8\pi^2}{g^2} \longrightarrow \frac{8\pi^2}{g^2} + \underbrace{\text{const}}_{\text{one loop contribution}}$$

In a SUSY gauge theory, β-function is

$$\beta = 3C(G) - \sum_r C_2(r)$$

For SU(N) with F fundamentals & antifundamentals

$$\beta = 3N - F$$

$$\frac{8\pi^2}{g^2(\mu)} = \frac{8\pi^2}{g^2(M)} + \underbrace{(3N-F)}_{b_0} \ln\frac{\mu}{M} \quad \longleftarrow \quad \text{holomorphic gauge coupling}$$

Only one loop renormalization. But we know that even in SUSY theory coupling renormalize beyond 1 loop. This is because kinetic terms renormalize

~~Physical scales~~

Given

$$\int d^4\theta \, Z(\mu)\left(\varphi_+^\dagger \varphi_+ + \varphi_-^\dagger \varphi_-\right)$$

Physical scale $\mu = \mu_0/Z(\mu)$ $M = M_0/Z(M)$

$$\frac{8\pi^2}{g^2(\mu)} - \underbrace{(3N-F)}_{b_0} \ln\mu = \text{const}$$

$\nearrow$ for holomorphic coupling. For physical scales in U(1) theories

$$\frac{8\pi^2}{g^2(\mu)} - b_0 \ln(\mu Z(\mu)) = \text{const}$$

Defining $\beta(g^2) = -\frac{g^4}{8\pi^2}\frac{\partial}{\partial \ln\mu} g^{-2}(\mu) = \frac{1}{8\pi^2}\frac{\partial}{\partial \ln\mu} g^2(\mu)$

and anomalous dimension $\gamma = -\frac{d}{d\ln\mu} \ln Z(\mu)$

we have for U(1)

$$\beta(g^2) = \frac{g^4}{16\pi^2} \sum T_F^i (1-\gamma^i)$$

In non-abelian gauge theories it is not straightforw. to introduce holomorphic regulator.

To generalize the beta-function, it is (probably) easiest to look at instanton amplitudes

$$\frac{8\pi^2}{g^2} - b_0 \ln\mu = \text{const}$$

$$\Lambda^{b_0} = \mu^{b_0} e^{-\frac{8\pi^2}{g^2(\mu)}} \longrightarrow \text{RG invariant}$$

including wave-function renorm. factor

$$\Lambda^{b_0} = \mu^{b_0} \prod_i Z(\mu)^{T_F^i} \frac{1}{(g^2(\mu))^{C_A}} e^{-8\pi^2/g^2(\mu)}$$

$$\beta(g^2) = \frac{g^4}{8\pi^2} \frac{3C_A - \sum T_F^i (1-\gamma^i)}{1 - (C_A g^2/8\pi^2)^2}$$

D-term non-renormalization:

$$\int d^4\theta \, \xi(m, g, \lambda) V$$

only is renormalized at one loop.