Introduction to Extra Dimensions

1 Introduction:

History: Nordström 1914: 5-dim vector theory
           \[ \Rightarrow \text{EM + scalar gravity} \]

After general relativity

Kaluza 1919, Klein 1926:

5-dim Einstein theory with one spatial dimension
compactified on a circle \[ \Rightarrow 4\text{-dim gravity + EM} \]

Late 1970's and 1980's: Higher dimensional theories

re-generate interests because of the developments in supergravity
and superstring theories. The consistency of superstring
theory requires extra dimensions. However, the extra
dimensions considered there are extremely small, \( \sim M_{\text{Pl}}^{-1} \)
beyond any experimental reach.

1990's: People started to consider extra dimensions

larger than \( M_{\text{Pl}} \)

- I. Antoniadis 1990: TeV\(^{-1}\) size extra dimension related to
  SUSY breaking

- Horava & Witten 1996: Extra dimension \( \sim (10^{12}\text{GeV})^{-1} \) in
  M-theory can lower the string scale to \( M_{\text{GUT}} \sim 10^{16}\text{GeV} \)

 Development of D-branes in string theory (Polchinski):

provides a natural setting for different fields living
in different number of extra dimensions, e.g.,
SM fields can be described by open strings which are
localized on lower-dimensional D-branes, while gravitons
are described by closed strings which propagate in
all dimensions.
The idea of extra dimensions became popular in phenomenology after Arkani-Hamed, Dimopoulos, and Dvali (ADD) 1998 considered large extra dimensions as a solution to the hierarchy problem.

Warped extra dimensions (Randall & Sundrum 1999) and AdS/CFT Correspondence (Maldacena 1998) provide new exciting possibilities to understand and construct models related to the weak scale. They will be covered in Tony Gherghetta's lecture, so we will focus on flat extra dimensions here.

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**Large Extra Dimensions as a Solution to the Hierarchy Problem**

(Arkani-Hamed, Dimopoulos, Dvali 1998)

**Hierarchy Problem** \( M_{pl} \gg M_{EW} \)

\( \sim 10^{19} \text{GeV} \quad \sim 10^3 - 10^4 \text{GeV} \)

The scale of quantum gravity is thought to be a possible scale to cut off the (quadratically) divergent radiative corrections in the Standard Model.

One can turn this question around: why is gravity so weak compared with other interactions in the SM?

Possibility: 4-dim. \( M_{pl} \) may not be a fundamental scale (or quantum gravity scale) if there exist large extra dimensions.
Newton's law in 4+n dimensions

\[ F(r) \sim \frac{G_{\text{N}}}{r^{n+2}} \frac{m_1 m_2}{M_{\text{pl}}(4+n)} \]

If \( n \) extra dimensions are compact with size \( L = 2\pi R \) then the force lines from a source mass have to go parallel in extra dimensions when the distance is larger than \( L \)

\[ F(r) \sim \frac{m_1 m_2}{M_{\text{pl}}(4+n)} \frac{1}{r^{n+2}} \quad \text{for} \quad r \ll L \]

\[ F(r) \sim \frac{m_1 m_2}{M_{\text{pl}}(4+n)} \frac{1}{L^n r^2} \quad \text{for} \quad r \gg L \]

Compared with 4D Newton's law

\[ F(r) \sim \frac{m_1 m_2}{M_{\text{pl}}^2} \frac{1}{r^2} \]

we have

\[ M_{\text{pl}}^2 (4+n) = M_{\text{pl}} (4+n) L^n = M_{\text{pl}} (4+n) V_n \]

where \( V_n \) is the volume of the compact extra dimensions

If we take the fundamental scale \( M_{\text{pl}}(4+n) \sim 1 \text{ TeV} \)

\[ M_{\text{pl}} (4) \sim 10^{19} \text{ GeV} \]

\[ \Rightarrow L \sim \left( \frac{M_{\text{pl}}^2 (4)}{M_{\text{pl}}^{n+2} (4+n)} \right)^{-\frac{1}{n}} \sim 10^{33n} \text{ TeV}^{-1} \]

\[ \sim 10^{33n} \times 10^{-17} \text{ cm} \]

Assuming extra dimensions have the same size
\[ n = 1 \Rightarrow L \sim 10^{15} \text{ cm} \quad (> 1 \text{ AU}) \quad \text{ruled out} \]
\[ n = 2 \Rightarrow L \sim 1 \text{ mm} \quad \text{(allowed in 1998, now } L \leq 200 \mu\text{m}) \]
\[ n = 3 \Rightarrow L \sim 10^{-6} \text{ cm} \]

On the other hand, SM has been tested up to a few hundred GeV \( \sim \) TeV \( \Rightarrow \) SM fields cannot propagate in extra dimensions with size \( R \gtrsim 1 \text{ TeV}^{-1} \), so SM fields have to be localized on a 3-brane (with width \( \lesssim 1 \text{ TeV}^{-1} \)).
Consider one extra dimension compactified on a circle with radius $R$
coordinates $x^0, x^1, x^2, x^3, x^5 = y$.

$M=0, 1, 2, 3, 5$ \quad $\mu = 0, 1, 2, 3$

**Ex1**: A real scalar field in extra dimensions

$$S = \int d^5x \frac{1}{2} \sum_{\mu \neq \nu} (x^\mu, y) \partial^\mu \Phi (x^\nu, y)$$

$\Phi$ : dim$ = \frac{3}{2}$

$\Phi (x^\mu, y + 2\pi R) = \Phi (x^\mu, y)$

Fourier decomposition of $\Phi$ along $y$ direction:

$$\Phi (x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \phi^{(n)}(x^\mu) e^{i\frac{n}{R} y}$$

$\Phi$ is real $\Rightarrow \phi^{(n)} = \phi^{(n)*}$

$$S = \int d^5x \frac{1}{2} \sum_{m, n} \left[ \frac{1}{2} \partial^\mu \phi^{(m)}(x^\mu) e^{i\frac{m}{R} y} \partial^\nu \phi^{(n)}(x^\nu) e^{i\frac{n}{R} y} 
- \frac{1}{2} (i\frac{m}{R}) \partial^\mu \phi^{(m)}(x^\mu) e^{i\frac{m}{R} y} (i\frac{n}{R}) \phi^{(n)}(x^\nu) e^{i\frac{n}{R} y} \right]$$

$$= \int d^5x \frac{1}{2} \sum_{m, n} \left[ \frac{1}{2} \partial^\mu \phi^{(m)}(x^\mu) \partial^\nu \phi^{(n)}(x^\nu) + \frac{m n}{R^2} \phi^{(m)}(x^\mu) \phi^{(n)}(x^\nu) \right]$$

$$= \int d^5x \frac{1}{2} \left[ \frac{1}{n} \partial^\mu \phi^{(-n)}(x^\mu) \partial^\nu \phi^{(n)} - \frac{n^2}{R^2} \phi^{(-n)}(x^\mu) \phi^{(n)} \right]$$

$$= \int d^5x \left[ \frac{1}{2} \partial^\mu \phi^{(m)} \partial^\nu \phi^{(m)} + \sum_{n=1}^{\infty} \left[ \partial^\mu \phi^{(m)} \partial^\nu \phi^{(n)} - \frac{n^2}{R^2} \phi^{(m)} \phi^{(n)} \right] \right]$$
From the 4D point of view it describes an (infinite) series of particles with masses \( m_{(n)} = \frac{n}{R} \) (Kaluza-Klein tower).

* If \( \Phi(x^n, y) \) has a 5D mass \( m_0 \), then the 4D Kaluza-Klein particles have masses \( m_{(n)}^2 = m_0^2 + \frac{n^2}{R^2} \).

* Generalization to higher extra dimensions compactified on a torus: \( m_{(n_5, n_6)}^2 = m_0^2 + \frac{n_5^2}{R_5^2} + \frac{n_6^2}{R_6^2} + \cdots \).

Ex2: Gauge field in 5D \( A_5(x^n, y) \)

\[
A_5(x^n, y) = \frac{1}{\sqrt{2\pi R}} \sum_n A_5^{(n)}(x^n) e^{-i\frac{n}{R} y}
\]

\( a_5 \rightarrow i\frac{n}{R} \)

\[
S = \int d^4x \, dy \left[ -\frac{i}{4} F_{MN} F^{MN} \right]
\]

\[
= \int d^4x \, dy \left[ -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} F_{\mu\nu} F^{\mu\nu} \right]
\]

\[
= \int d^4x \, dy \left[ -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} (\partial_{\mu} A_5 - \partial_{\nu} A_5) (\partial^{\mu} A_5 - \partial^{\nu} A_5) \right]
\]

\[
= \int d^4x \, \frac{2}{n} \left[ -\frac{i}{4} F_{\mu\nu}^{(n)} F^{(n)}_{\mu\nu} + \frac{i}{2} (\partial_{\mu} A_5^{(n)} + i\frac{n}{R} A_5^{(n)}) (\partial^{\mu} A_5^{(n)} - i\frac{n}{R} A_5^{(n)}) \right]
\]

For \( n \neq 0 \) we can perform the gauge transformation,

\[
A_5^{(n)} \rightarrow A_5^{(n)} + i\frac{1}{nR} \partial_{\nu} A_5^{(n)} \quad (F_{\mu\nu}^{(n)} \rightarrow F_{\mu\nu}^{(n)'}).
\]
\[ S = \int d^4x \left\{ \left( -\frac{1}{4} F_{\mu \nu}^{(n)} F^{(n) \mu \nu} + \frac{i}{2} \partial_\mu A_5^{(n)} \partial^\mu A_5^{(n)} \right) + \sum_{n=1}^{\infty} 2 \left[ -\frac{1}{4} F_{\mu \nu}^{(-n)} F^{(-n) \mu \nu} + \frac{i}{2} \frac{n^2}{R^2} A_\mu^{(-n)} A_\mu^{(-n)} \right] \right\} \]

Zero mode: 4D gauge field + 1 scalar (in adjoint rep if coming from ren-Abelian)

Non-zero mode: \( A_5^{(n)} \) is eaten and becomes the longitudinal mode of massive vector field \( A_\mu^{(n)} \) of mass \( \frac{m}{R} \)

* 5D covariant derivative \( D_\mu = \partial_\mu + i g_5 A_\mu \)

\[ D_\mu = \partial_\mu + i g_5 A_\mu = \partial_\mu + i g_5 \frac{1}{\sqrt{2\pi R}} A_\mu^{(0)} + \ldots \]

\( A_\mu : \text{dim } \frac{3}{2} \Rightarrow g_5 : \text{dim } -\frac{1}{2} \)

related to 4D gauge coupling by

\[ g_4 = \frac{g_5}{\sqrt{2\pi R}} \]

* 5D gauge theory is non-renormalizable. It becomes strongly interacting at energy scale \( E \sim \frac{1}{g_4^2} = \frac{1}{2\pi R g_4^2} \)

It can only be treated as a low energy effective theory with a cutoff \( \Lambda \sim \frac{4\pi}{g_5^2} \). From the 4D point of view, the strong interaction comes from the number of KK modes accessible at the energy scale. Effective coupling: \( \sqrt{N_k} g_4 \)

* In higher (4+n) dimensions

Zero mode: 4D gauge field + n (adjoint) scalars

Non-zero modes: 4D massive vector field + (n-1) massive (adjoint) scalars
Ex 3: Gravitational field in $D = 4 + n$ dimensions

described by a (symmetric) metric tensor $g_{MN} = \eta_{MN} + h_{MN}$

There are $\frac{D(D+1)}{2}$ independent components of $h_{MN}$

Gauge symmetry: $D$-dimensional general coordinate transformation

$h_{MN} \rightarrow h_{MN} + \omega_M^N \xi_N + \omega_N^M \xi_M$

We can impose $D$ conditions to fix the gauge.

e.g., harmonic gauge: $\omega_M h_N^M = \frac{1}{2} \omega_N h_M^M$

However, gauge transformations satisfying $\Box \xi_M = 0$ are still allowed. Another $D$ conditions can be imposed.

$\Rightarrow$ # of independent degrees of freedom

$\frac{D(D+1)}{2} - 2D = \frac{D(D-3)}{2}$

$D = 4 \Rightarrow 2$

$D = 5 \Rightarrow 5$

$D = 6 \Rightarrow 9$

A 4D massive spin-2 field has 5 polarization

[Mass term of a spin-2 field $a h^{\mu\nu} + b h^2$ $h = h^\mu_{\mu}$]

In general, there are 6 d.o.f. but one of them is a ghost. The ghost is removed for $a = -b$ (Fierz-Pauli)

then only 5 d.o.f are left
5D graviton with one spatial dimension compactified

\[ h_{MN} = h_{\mu\nu} \otimes h_{\rho\sigma} \otimes h_{\gamma\delta} \]

**Zero mode:** 1 4D graviton + 1 massless vector + 1 real scalar

(Kepler–Klein's attempt to unify gravity & EM into 5D GR)

**Non-zero modes:** \( h_{\mu\nu}^{(n)}, h_{\rho\sigma}^{(n)} \) are eaten by \( h_{\mu\nu}^{(n)} \) to form a massive spin-2 field.

4+n dimensional graviton reduced down to 4 dimensions

**Zero mode:** 1 4D graviton, \( n \) massless vector, \( \frac{n(n+1)}{2} \) scalars

**Non-zero modes:** 1 massive spin-2 tensor, \( (n-1) \) massive spin-1 vectors,

\[
\frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2} \text{ massive scalars}
\]

\[ \uparrow \quad \uparrow \]

\[ \text{eaten by spin-2} \quad \text{spin-1} \]

**Relations between the (reduced) Planck scales in 4 dimensions and higher dimensions**

\[ S = \frac{M_{4+n}}{2} \int d^{4+n}x \sqrt{|g_{4+n}|} R_{4+n} \]

\[ = \frac{M_{4+n}}{2} (2\pi R)^n \int d^4x \sqrt{-g_4} R_4 + \ldots \]

\[ = \frac{\overline{M}_4^{2+n}}{2} \int d^4x \sqrt{-g_4} R_4 + \ldots \]

\[ \Rightarrow \overline{M}_4^2 = (2\pi R)^n \overline{M}_{4+n}^{2+n} \]

**Reduced Planck scale**

\[ \overline{M}_4 = \frac{1}{18\pi^2 G_N} = \frac{M_{pl}}{\sqrt{8\pi}} = 2.4 \times 10^{18} \text{ GeV} \]
The 4D graviton and its KK modes

Spin-2:

\[ \begin{pmatrix}
\begin{array}{c}
\Gamma_{mn}^i \\
\Gamma_{mn}^{ij}
\end{array}
\end{pmatrix} \]

\[ \mathbf{k} = (k_0, k_5, \ldots, k_3) \]

: KK numbers along extra dim

10 components - 5 gauge conditions = 5 d.o.f

\[ \partial^\mu \Gamma_{\mu
\nu}^i = 0 \]

\[ \Gamma_{\mu
\nu}^i = 0 \]

Spin-1:

\[ \begin{pmatrix}
\begin{array}{c}
V_{ij}^k \\
V_{ij}^{ik}
\end{array}
\end{pmatrix} \]

: \( n-1 \) massive vector

\[ \mathbf{k} \cdot V_{ij}^k = 0 \]

Lorentz gauge condition \( \partial^\mu V_{ij}^k = 0 \) can be imposed

\[ \Rightarrow \] a physical d.o.f for each massive vector

Spin-0

\[ \begin{pmatrix}
\begin{array}{c}
S_{ij}^k \\
S_{ij}^{ik}
\end{array}
\end{pmatrix} \]

\[ \mathbf{k} \cdot S_{ij}^k = 0 \]

\[ \Rightarrow \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} \] scalars

We usually separate out the radion field \( h_{ij}^k \), so we can impose an additional condition on the rest of the scalars

\[ S_{ij}^k = 0 \]
Canonically normalized 4D fields in unitary gauge are given by

Radion: \[ H_j^\xi = \frac{1}{\alpha} h_j^\xi j \quad \alpha = \sqrt{\frac{3(n-1)}{n+2}} \]

Other scalars: \[ S_{ij}^\xi = h_{ij}^\xi - \frac{\alpha}{n-1} \left( \eta_{ij} + \frac{\hat{H}_i^\xi \hat{H}_j^\xi}{k^2} \right) H_k^\xi \]

Vectors: \[ V_{ij}^\xi = \frac{1}{\sqrt{2}} h_{ij}^\xi \]

Gravitons: \[ G_{\mu\nu}^\xi = h_{\mu\nu}^\xi + \frac{\alpha}{3} \left( \eta_{\mu\nu} + \frac{\partial \phi}{k^2} \right) H_{\mu\nu}^\xi \]

The equation of motion in the presence of sources on a 3-brane

\[
(\square + \hat{k}^2) \begin{pmatrix} G_{\mu\nu}^\xi \\ V_{ij}^\xi \\ S_{ij}^\xi \\ H_k^\xi \end{pmatrix} = \begin{pmatrix} -\frac{1}{M_p} \left( -\eta_{\mu\nu} + (\eta_{\mu\nu} + \frac{\partial \phi}{k^2}) \right) \frac{T_{\mu\nu}^A}{3} \\ 0 \\ 0 \\ \frac{\alpha}{3 M_p} \frac{T^A}{T^A} \end{pmatrix}
\]
Effective Field Theory for a 3-brane

(R. Sundrum, hep-ph/9805471)

Assume that SM fields live on a 3-brane in a higher-dimensional space-time. How do we write down an EFT for that

3-brane can spontaneously or explicitly break the higher-dim space-time symmetry

Ex. spontaneous: (domain walls, D-branes) ⇒ Goldstone bosons
explicitly: (boundaries orientifolds)
We will describe spontaneous breaking here

Notations: assume that 3-brane = $\mathbb{R}^4$, extra dimensions = $T^n$ (torus)
coordinates in the bulk $X^M$, $M = 0, 1, \ldots, 3+n$

" on the brane $x^\mu$, $\mu = 0, 1, 2, 3$

" along the extra-dim $X^m$, $m = 4, \ldots, 3+n$

Metric in 4+n dimensions: $G_{MN}(X)$
The bulk coordinates describing the position occupied by a point $x^\mu$ on the 3-brane are denoted $Y^M(x)$
and they are dynamical fields

* SM fields are functions of $x^\mu$: $\phi(x), A_\mu(x), \psi(x)$

The EFT describes small fluctuations around the vacuum state

Vacuum: $G_{MN}(X) = \eta_{MN}$ (signature $+---\ldots$)

$Y^M(x) = \delta^M_N x^N$ (a simple gauge choice)
Bulk action: \( S_{\text{bulk}} = \int d^{d+1}x \sqrt{|g|} \left( \frac{M_0^2}{2} R - \Lambda \right) \)

To write down an effective action for the brane localized fields, we need the induced metric on the brane,
\[
ds^2 = G_{MN} \frac{d\Gamma^M(x)}{dx^M} \frac{d\Gamma^N(x)}{dx^N} = G_{MN} \frac{\partial \Gamma^M}{\partial x^\mu} \frac{\partial \Gamma^N}{\partial x^\nu} dx^\mu dx^\nu
\]

So \( g_{\mu \nu} = G_{MN} \frac{\partial \Gamma^M}{\partial x^\mu} \frac{\partial \Gamma^N}{\partial x^\nu} \)

In the vacuum state \( g_{\mu \nu} = \eta_{\mu \nu} \)

The action is invariant under:

- General coordinate transformations of bulk coordinates \( x^M \)
- "brane" \( x^\mu \)

so we need to contract the indices \( M \) and \( \mu \) separately

\[
S_{\text{brane}} = \int d^d x \sqrt{|g|} \left\{ \frac{\kappa_0^2}{2} R(x) - \frac{1}{2} g_{\mu \nu} \partial \phi \partial \phi - V(\phi) + \frac{g^\mu_\nu g_{\rho \sigma}}{4} F_{\mu \nu} F_{\rho \sigma} + \cdots \right\}
\]

\( M_0 \) receives contributions from both the bulk term \( M_0^2 \) and the brane term \( \kappa_0^2 \)

Assume the brane tension \( f \ll M_0 \) (\( \ll \kappa_0 \)), then we can ignore the back reaction on gravity.
We can use 4D reparametrization invariance to gauge fix
\[ Y^m(x) = x^m \quad (m = 0, 1, 2, 3) \]

⇒ Only \( Y^m(x) \), \( m = 4, 5, \ldots n+3 \) are physical degrees of freedom. Their kinetic terms are given by
\[
S = \int d^4x \sqrt{|g|} \left[ -f^4 \cdots \right]
\]
\[
\delta g^\mu \nu = \epsilon^{mn} \varepsilon_r Y^m \varepsilon^r Y^n = g^\mu \nu + \varepsilon_r Y^m \varepsilon^r Y^n + \cdots
\]
\[
\det g = -1 - \varepsilon_r Y^m \varepsilon^r Y^n + \cdots
\]
\[
\sqrt{|g|} = 1 + \frac{1}{2} \varepsilon_r Y^m \varepsilon^r Y^n
\]
\[
S = \int d^4x \left( -f^4 \right) \left( 1 + \frac{1}{2} \varepsilon_r Y^m \varepsilon^r Y^n + \cdots \right)
\]
\[
= \int d^4x \left[ (-f^4) + \frac{f^4}{2} \varepsilon_r Y^m \varepsilon^r Y^n + \cdots \right] \quad (Y^m = -Y^n)
\]

Canonically normalized fields are given by \( Z^m = f^4 Y^m \)
Positive tension \( (f^4 > 0) \) ⇒ positive kinetic term
Negative tension \( (f^4 < 0) \) ⇒ negative kinetic term
⇒ \( Y^m \) are ghost ⇒ instability
( The brane will crumble )

(Negative tension can happen with explicit higher-dim space-time
symmetry breaking where there is no NG bosons \( Y^m \),
e.g., orientifold )
Counting of SM fields to bulk gravitons.

\[ T^{\mu \nu} = \frac{2 \delta S}{\sqrt{g}} \delta g_{\mu \nu} \]

\[ S_{\text{int}} > \int d^4x \sqrt{|g|} \frac{1}{2} T^{\mu \nu} \delta g_{\mu \nu} \]

\[ g_{\mu \nu} = g_{\mu \nu}^{(0)} + \sum_{n} \gamma_{\mu} \otimes \gamma_{\nu}^{(n)} \]

\[ = (\gamma_{\mu \nu} + K_{4+n} H_{\mu \nu}) (\delta_{\mu}^{(0)} + \cdots) (\delta_{\nu}^{(N)} + \cdots) \]

\[ = \gamma_{\mu \nu} + \frac{2}{M_{4+n}} H_{\mu \nu} \]

For extra dim compactified on a torus, \( H_{\mu \nu} (x^2) \) is periodic

\[
H_{\mu \nu} (x, y) = \sum_{k_{1}} \cdots \sum_{k_{n}} \frac{e^{i k_{1} y_{1}}}{\sqrt{V}} \frac{e^{i k_{2} y_{2}}}{\sqrt{V}} \cdots \frac{e^{i k_{n} y_{n}}}{\sqrt{V}} \]

We can choose \( \vec{y} = 0 \) for the 3-brane location

\[ H_{\mu \nu} (x, 0) = \frac{2}{F} \frac{h_{\mu \nu}^{(0)}}{V} \]

\[ S_{\text{int}} > \int d^4x \sqrt{|g|} \frac{1}{2} T^{\mu \nu} \sum_{k} \frac{2}{M_{4+n} \sqrt{V}} h_{\mu \nu}^{(k)} \]

Ref: Giudice, Rattazzi, Wells hep-ph/9811291

Han, Lykken, Zhang hep-ph/9811350