VII Boundary conditions and orbifolds

We discussed Kaluza-Klein theory for an extra dimension compactified on a circle (or a torus for more dimensions).

Now consider a (scalar) field $\Phi$ living in an extra dim which is a line segment $0 \rightarrow \pi R$.

The current in $y$ direction is $J_y = \bar{\Phi} \gamma^3 \gamma^5 \Phi$. To preserve unitary we should have $J_y = 0$ at the boundaries, so we need $\Phi = 0$ or $\partial_y \Phi = 0$ at the boundaries.

$\partial_y \Phi |_{\text{boundary}} = 0$ : Neumann boundary condition

$\Phi |_{\text{boundary}} = 0$ : Dirichlet

Neumann BC. : $\partial_y \Phi(y=0) = \partial_y \Phi(y=\pi R) = 0$

$$\Phi_+(x, y) = \frac{1}{\pi R} \Phi^{(0)}_+(x) + \frac{\sqrt{2}}{\pi R} \sum_{n=1}^{\infty} \phi_+^{(n)}(x) \cos \frac{n y}{R} \quad m_n = \frac{n}{R}$$

Dirichlet BC. : $\Phi(y=0) = \Phi(y=\pi R) = 0$

$$\Phi_-(x, y) = \frac{\sqrt{2}}{\sqrt{2} \pi R} \sum_{n=1}^{\infty} \phi_-^{(n)}(x) \sin \frac{n y}{R} \quad m_n = \frac{n}{R}$$

There is no zero mode for Dirichlet BC!
The line segment can also be described as an orbifold $S^{1}/\mathbb{Z}_2$.

Torus compactification ($S'$): identifying $\Phi(y+2\pi n)$ and $\Phi(y)$, or equivalently, "gauging" a discrete subgroup $g \rightarrow g + 2\pi n$ of translation in $y$ direction.

If a theory has an exact global symmetry, we can mod out (gauge) a subgroup of the global symmetry and still obtain a consistent theory.

Orbifold: a space obtained by modding out a symmetry transformation of another space which leaves some points fixed.

E.g. $S^{1}/\mathbb{Z}_2$

\[ \begin{array}{c}
\mathbb{Z}_2 \text{ symmetry } f \rightarrow -f \\
S^{1}/\mathbb{Z}_2 \text{ identifying } y \text{ and } -y \\
\text{Fixed points: } y=0, y=\pi R
\end{array} \]
Orbifold projection requires \( y \to -y \) to be a good symmetry in the original theory \( \Rightarrow \) Fields defined on \( (-\pi R, \pi R) \) can be categorized as even or odd under \( y \to -y \)

Even \( \Phi^+ : \Phi^+(-y) = \Phi^+(y) \Rightarrow \Phi^+_z(y=0) = 0 = \Phi^+_z(y=\pi R) \) "Neumann BC"

Odd \( \Phi^- : \Phi^-(y) = -\Phi^-(y) \Rightarrow \Phi^-_z(y=0) = 0 = \Phi^-_z(y=\pi R) \) "Dirichlet BC"

Gauge fields \( A_\mu(x, y) \)

\( \mathbb{Z}_2 : x_m \to x_m, \ y \to -y \Rightarrow \Phi^+ \) even, \( \Phi^- \) odd

\( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) needs to have a definite parity

\( \Rightarrow A_\mu \) and \( A_\nu \) have opposite parity

If \( A_\mu = \text{even} \) then \( A_\nu = \text{odd} \) ad has no zero mode

\( \Rightarrow \) no extra light scalar

Fermions

\( \Gamma^M = (\Gamma^a, \Gamma^5) \quad \Gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix} \quad \sigma^a = (1, \sigma) \quad \overline{\sigma}^a = (1, -\sigma) \)

\( \Psi = \begin{pmatrix} \chi \\ \bar{\Psi} \end{pmatrix} \quad \overline{\Psi} = \begin{pmatrix} \bar{\chi} \\ \Psi \end{pmatrix} \)

\( S = \int d^4x \ d\bar{\Psi} i \Gamma^M \partial_\mu \Psi \)

\( = \int d^4x \ d\bar{\Psi} i \overline{\chi} \sigma^a \partial_\mu \chi + i \bar{\Psi} \sigma^a \partial_\mu \chi + 4 \bar{\Psi} \sigma^5 \chi - \bar{\chi} \sigma^a \partial_\mu \Psi \)

\( \partial_5 = \text{odd} \Rightarrow \Psi, \bar{\Psi} \) have opposite parity

\( \Rightarrow \) only one \( i \) even ad has a zero mode

\( \Rightarrow \) zero mode is chiral
This is also consistent with gauge symmetry
\[ \Phi^{\dagger} \Gamma A \Phi = \bar{\chi} \delta_{\mu} A_{\mu} \chi + \frac{i}{2} \sigma^{\mu} A_{\mu} \Phi \gamma + \gamma A_{\mu} \chi - \bar{\chi} A_{\mu} \Phi \]

Question: Is mass term allowed?
\[ m \Phi \Phi = m (4 \chi + \bar{\chi} \Phi) \]
m must be odd in orbifold language. It's ok since the theory is really just defined on one line segment

* Universal Extra Dimensions (UEDs):*

Since SM fermions are chiral, they can live in extra dimensions if orbifold compactification or boundary conditions can give rise to chiral zero mode. In this case, we can have all SM fields propagate in some extra dimensions

- UEDs have weaker bounds from EW precision data
  - Approximate KK number conservation if Lagrangian terms localized on boundaries (orbifold fixed points) are ignored

Tree-level contributions to EW observables from KK states are suppressed, they can contribute at one-loop level
\[ \sqrt{s} (\text{UED}) \gtrsim 500 \text{ GeV} \]
- Boundary terms will be present as they are not forbidden by symmetries and they are induced by bulk loop corrections. They modify the KK spectrum from $m_{n+1} = \frac{n}{R}$ and lift the degeneracies of KK excitations of different SM species.

- Some discrete subgroup of KK number can still be preserved.

Eq. $\mathbb{Z}_2$

A $\mathbb{Z}_2$ reflection about $y = \frac{\pi R}{2}$ is a good symmetry if the boundary terms at $y=0$ and $y=\pi R$ are equal \( \Rightarrow \) KK parity. It implies that the lightest first KK excitation of all SM fields is stable, and can be a good dark matter candidate (KK dark matter).
*Symmetry breaking by orbifolds (boundary conditions)*

When there is a symmetry of the Lagrangian, the fields at the identified points don't have to be identical, but merely equal up to a symmetry transformation.

[Instead of modding out a discrete subgroup of the space-time symmetry, one mod out a diagonal combination of the space-time symmetry and the internal symmetry.]

E.g., GUT breaking \([SU(5) \rightarrow SU(3) \times SU(2) \times U(1)]\) on \(S^1/\mathbb{Z}_2\)

\[
SU(5) : \begin{pmatrix}
SU(3) & x, y \\
\cdots & \cdots \\
x, y & SU(3)
\end{pmatrix} \quad U(1) \sim \begin{pmatrix}
2 & 2 & 2 \\
-3 & -3 & -3
\end{pmatrix}
\]

Consider the \(\mathbb{Z}_2\) subgroup of \(SU(5)\), \(\exp[\lambda (\begin{pmatrix}1 & 1 & 1 \end{pmatrix})^T]\)

Fundamental \(5 \rightarrow \begin{pmatrix}+ & + & - & - & - & - & - & - & - \end{pmatrix}
\]

Adjunct \(24 \rightarrow \begin{pmatrix}+ & + & - & - & - & - & - & - & - \end{pmatrix}
\]

Choose the \(\mathbb{Z}_2\) of \(S^1/\mathbb{Z}_2\) to be the diagonal combination of \(\mathbb{Z}_2\) subgroup of \(SU(3)\) and \(y \rightarrow -y\)

\[
SU(3) \times SU(2) \times U(1) \quad x, y
\]

\[
A_4 \quad + \\
A_5 \quad -
\]

Only gauge fields corresponding to \(SU(3) \times SU(2) \times U(1)\) have zero modes \(\Rightarrow SU(5)\) is broken down to \(SU(3) \times SU(2) \times U(1)\)
This can be described equivalently by boundary conditions

\[ SU(3) \times SU(2) \times U(1) \quad \text{X, Y} \]

\[ A_n \quad (N, N) \quad (D, D) \]

\[ A_s \quad (D, D) \quad (N, N) \]

\[ \text{BC at } y = 0 \quad \text{BC at } y = \pi \]

\[ SU(3) \times SU(2) \times U(1) \quad \text{SU}(5) \]

\[ \text{SU}(5) \quad \text{SU}(3) \times SU(2) \times U(1) \]

\[ \text{SU}(3) \times SU(2) \times U(1) \quad \text{X, Y} \]

\[ A_n \quad (N, N) \quad (N, D) \]

\[ A_s \quad (D, D) \quad (D, N) \]

\[ \text{SU}(5) \]

\[ \text{SU}(5) \quad \text{SU}(3) \times SU(2) \times U(1) \]

\[ \text{SU}(3) \times SU(2) \times U(1) \]

(This corresponds to \( S^1 \times Z \) orbifold)

A_5's of X, Y have zero modes \( \Leftrightarrow \) light charged scalars

(They obtain masses from finite one loop correction, \( m = \frac{g^2}{4\pi} \))
More general BC's:

Neumann + boundary terms

\[ S_{\text{bulk}} = \int d^4x \int_0^{\pi R} dy \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \]

\[ \delta S_{\text{bulk}} = \int d^4x \int_0^{\pi R} dy \left[ \partial_\mu \phi \partial^\mu \delta \phi - \frac{\partial V}{\partial \phi} \delta \phi \right] \]

\[ = \int d^4x \int_0^{\pi R} dy \left( -\partial_\mu \phi \partial^\mu \phi - \frac{\partial V}{\partial \phi} \right) \delta \phi = \int d^4x \partial_\phi \delta \phi \bigg|_0^{\pi R} \]

For \( \delta S_{\text{bulk}} = 0 \) we need

\[ \partial_\mu \partial^\mu \phi = -\frac{\partial V}{\partial \phi} \quad \text{equation of motion} \]

and

\[ \partial_\phi \delta \phi \bigg|_{\text{boundary}} = 0 \]

which can be satisfied by Neumann B.C. \( \partial_\phi \delta \phi \bigg|_{\text{boundary}} = 0 \)

for arbitrary \( \delta \phi \bigg|_{\text{boundary}} \)

Other boundary conditions can be obtained by adding boundary terms.

\[ S = S_{\text{bulk}} - \int d^4x \frac{1}{2} M_1^2 \phi^2 \bigg|_{y=0} - \int d^4x \frac{1}{2} M_2^2 \phi^2 \bigg|_{y=\pi R} \]

\[ \delta S = \int d^4x \left( \text{EOM} \right) \delta \phi \]

\[ -\int d^4x \left( \partial_\phi + M_1^2 \phi \right) \delta \phi \bigg|_{y=\pi R} + \int d^4x \left( \partial_\phi - M_2^2 \phi \right) \delta \phi \bigg|_{y=0} \]

BC's

\[ \partial_\phi + M_1^2 \phi = 0 \quad \text{at} \quad y = \pi R \]

\[ \partial_\phi - M_2^2 \phi = 0 \quad \text{at} \quad y = 0 \]

For \( M_1, M_2 \to \infty \) \( \phi = 0 \quad \text{at} \quad y = 0, \pi R \)

and \( \partial_\phi \) arbitrary at \( y = \varepsilon, \pi R - \varepsilon \)

This is equivalent to Dirichlet BC's.
Similarly for a gauge field
\[ \int d^{4}x \int_{\phi}^{\phi} d\phi \left( -\frac{1}{4} F_{\mu\nu}^{a} F^{a\mu\nu} \right) \]

**Natural BC (for unitary gauge):**
\[
\partial_{\phi} A_{0}^{a} = 0 \quad A_{3}^{a} = 0
\]

Now we add boundary scalar fields which have nonzero vevs
\[ L_{i} = | D_{\mu} \Phi_{i} |^{2} - \lambda_{i} \left( | \Phi_{i} |^{2} - \frac{1}{2} V_{i}^{2} \right)^{2} \]
\[ i = 1, \; y = 0 \]
\[ i = 2, \; y = \pi R \]
\[ \Phi_{i} = \frac{1}{\sqrt{2}} \left( V_{i} + h_{i} \right) e^{i \pi / \lambda_{i}} \]

**Natural BC**
\[ \partial_{\phi} A_{0}^{a} + V_{i}^{2} A_{0}^{a} \left|_{\phi=0} \right. = 0 \]

Taking \( V_{i} \rightarrow \infty \Rightarrow A_{0} \left|_{\phi=0} \right. = 0 \) corresponds to Dirichlet BC.

\( h_{i}, \pi_{i} \) decouple from the gauge field in this limit.

At the same time, \( A_{3} \) BC changes from Dirichlet to Neumann \( \partial_{\phi} A_{3} = 0 \)

Mass of the lightest gauge boson \( \sim \frac{1}{R} \) independent of \( V_{i} \)

\[ \begin{array}{c}
\text{Gauge symmetry breaking } SU(2) \rightarrow U(1) : \\
A_{\mu}^{1,2}(y = 0) = 0, \; A_{\mu}^{1,2}(y = \pi R) = 0, \; \partial_{\phi} A_{\mu}^{3}(y = 0) = 0, \; \partial_{\phi} A_{\mu}^{3}(y = \pi R) = 0
\end{array} \]

equivalent to orbifold breaking

\[ SU(2) \rightarrow \text{nothing} \]
\[ A_{\mu}^{1,2}(y = 0) = 0, \; \partial_{\phi} A_{\mu}^{3}(y = 0) = 0, \; A_{\mu}^{3,3}(y = \pi R) = 0, \; \partial_{\phi} A_{\mu}^{3}(y = \pi R) = 0 \]
Dimension deconstruction

Gauge theories in more than 4 dimensions are non-renormalizable. They can only be treated as low energy effective theories below some cutoff $\Lambda$. However, one needs to be careful about how to implement the cutoff and asking cutoff-sensitive questions.

A naive momentum cutoff breaks gauge invariance. It also violates the locality in extra dimensions. One can easily get a nonsensical answer if one doesn't regularize the theory in a correct way. One way to regularize the higher dimensional gauge theories while preserving gauge invariance and locality is to put extra dimensions on a lattice.

Consider $N+1$ copies of $SU(N)$ gauge groups in 4D, with $N$ link-Higgs fields $\Phi_i$, which transform as bi-fundamentals $(\bar{N}_i, N_{i-1})$ under neighboring gauge groups.

$$L = -\frac{1}{4} \sum_{i=0}^{N} F_{i\mu\nu}^a F_{i\mu\nu}^a + \sum_{i=1}^{N} D^\mu \Phi_i^+ D^\nu \Phi_i$$

$$D^\mu = \partial^\mu + i \sqrt{\frac{N}{6}} A_i^\mu T^i$$

where $\sqrt{\frac{N}{6}}$ is the gauge coupling of $SU(N_i)$ group (assuming to be equal)
Assume that each $\Phi_i$ obtain a vacuum expectation value of the form through some potential (and all left over physical Higgses are heavy)

$$\langle \Phi_i \rangle = U \begin{pmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & 1 \end{pmatrix}$$

These vevs break $N+1$ SU($N$) gauge groups down to the diagonal SU($n$) gauge group. The mass matrix for the $N+1$ gauge fields $A^a_{\mu}$ ($i=0, \ldots, N$) is

$$M^2 = g^{-2} u^2 \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

The mass eigenstates $\tilde{A}_\mu^a$ can be obtained by diagonalizing the mass matrix

$$\tilde{A}_\mu^a = \frac{\sqrt{N}}{n=0} a_{jn} \tilde{A}_\mu^n$$

For $n \neq 0$

$$a_{jn} = \sqrt{\frac{2}{N+1}} \cos \left( \frac{3j+1}{2} \frac{\pi n}{N+1} \right) \quad j=0, 1, \ldots, N$$

$$\chi_n = \frac{\pi n}{N+1}$$

$$a_{j0} = \frac{1}{\sqrt{N+1}} \quad j=0, 1, \ldots, N$$

Mass for $\tilde{A}_\mu^a$ is given by

$$M_n = 2 g u \sin \left( \frac{\chi_n}{2} \right)$$
For small $n\ll N$

\[ M_n \approx \frac{\tilde{g} v \pi n}{N+1} \]

same as the KK tower from an extra dimension

To match on to the KK spectrum

\[ \frac{\tilde{g} v \pi}{N+1} = \frac{1}{R} = \frac{\pi}{L} \Rightarrow L = \frac{N+1}{\tilde{g} v} \]

(This corresponds to an orbifold compactification)

The low energy gauge coupling (of the unbroken diagonal gauge group) is given by

\[ g = \frac{\tilde{g}}{\sqrt{N+1}} \]

The 4D theory with $N$ SU($N$) gauge groups provide a UV (renormalizable) completion of 5D gauge theory with a cutoff

\[ \Lambda \sim \frac{1}{a} = \frac{1}{g v}, \quad a: \text{ lattice spacing} \]

Many results in higher-dimensional theories can be easily translated into 4D language

E.g. Orbifold symmetry breaking $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$

\[ SU(5) \rightarrow SU(3) \rightarrow SU(2) \rightarrow U(1) \]

"Theory space"

* Deconstructing gravity is much more difficult, however.