Fermions in Holographic Superconductors

In $p$-wave holographic superconductors, the Fermi surface degenerates to a pair of points, above each of which a Dirac cone rises, enclosing a continuum of fermion modes. There are also discrete fermion normal modes slightly outside the Dirac cones.
The underlying lagrangian on which all the calculations on this topic will be based is

\[ L = R - \frac{1}{2} \text{tr} F_{\mu \nu}^2 - i \psi \Gamma^\mu \partial_\mu \psi + \frac{6}{L^2} \]

in bulk spacetime dimension \( D = 4 \). Thus, the simplest solution to the eom's is

\[ \text{AdS}_4: \quad ds^2 = \frac{L^2}{z^2} \left[ -dt^2 + (dx^1)^2 + (dx^2)^2 + dz^2 \right] \]

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] \]

where \( A_\mu = A_\mu^a T^a \) and \( T^a = \frac{1}{2} \sigma^a \) are the generators of SU(2), \( a = 1, 2, 3 \)

\( \psi \) is a doublet of SU(2), and
\[ D_\mu \Psi = \left( \nabla_\mu - ig_{\mu \gamma} A^a_{\gamma} T^a \right) \Psi \]

includes spin connection

The $T \to 0$ limit of the p-wave holographic superconductor is an AdS$_4$ to AdS$_4$ domain wall (of the $T \to 0$ limit of RNAdS which interpolates between AdS$_4$ and AdS$_2 \times \mathbb{R}^2$).

\[ A_{\mu}^{\text{AdS}} \rightarrow A_{\mu}^{\text{IR}} \]

$A_{\mu}^{\text{UV}}$ and $A_{\mu}^{\text{IR}}$ are flat gauge connections; zero field strength.

The main results on fermion 2 pt functions
can be understood starting from the gauge-covariant wave vector:

\[ K_m \equiv k_m - g_{\gamma m} A_m = k_m \mathbb{1} - g_{\gamma m} A_m^a T^a \]

\[ K^m_\text{IR} = k_m - g_{\gamma m} A^\text{IR}_m \]

\[ K^{uv}_m = k_m - g_{\gamma m} A^{uv}_m \]

\( m = 0, 1, 2 \), and \( A^\text{IR}_z \equiv 0 \) by a gauge choice.

Then

\[ \langle \phi^+ \phi \rangle \sim G_{\text{sudden}}(k) \]

\[ \equiv - i (\gamma^m q_m)^{-1} (q + \gamma^m q_m q_n) j^+ \]

where \( q = K^\text{IR}_m \coth(z S_k) \)

\[ K^\text{IR}_m = \sqrt{\gamma^{mn} K^{\text{IR}m}_m K^{\text{IR}n}_n}, \quad K^{uv}_m = \sqrt{\gamma^{mn} K^{uv}_m K^{uv}_n} \]
Similar definitions for $q_m$ and $q_{mn}$.

The continuous part of the spectral weight of $C^\Psi_{\text{Sudden}}(k) \text{ arises precisely where } K^\text{IR}_m \text{ has a branch cut} \text{ — i.e. where } K^\text{IR}_m \text{ is timelike} — \text{ because otherwise, } C^\Psi_{\text{Sudden}}(k) \text{ is a rational function of the } k_m.

$K^\text{IR}_m \text{ is timelike inside the aforementioned Dirac cones.}$

- This topic is based on 1002.4416 with J. Rocha and A. Yarom.

The motivations were numerous:
• We knew about holographic superconductors, both S-wave & P-wave.

• We knew about fermion correlators in the normal state (see H. Lüüs lectures, for example).

• To have some chance at successful comparison to ARPES, where Dirac cones above isolated points on the Fermi surface are observed, we knew we needed non-S-wave dynamics.

• The lagrangian we chose is almost completely determined (at 2-derivative level) by its symmetries: basically
its QCD with $N_c = 2$ & $N_f = 1$

\[-\frac{1}{2} \pi F^2 - i \overline{\Phi} \Phi \Phi\]
coupled to gravity with a c.c. \((R + \frac{6}{L^2})\).

- It's easy to get similar lagrangians out of string/M-theory low-energy effective actions.

- Basu et al.'s 0810.3970 had recently explained the AdS$_4$-to-AdS$_4$ domain wall structure.

Holographic superconductors

The main macroscopic features of superconductors are a consequence of the spontaneous...
breaking of $U(1)_{EM}$, at finite $T$ and finite chemical potential $\mu$:

$$
\begin{array}{c}
T \\
\uparrow \\
AF \\
\downarrow \\
SC \\
\uparrow \\
SM \\
PG \\
\downarrow \\
FL \\
\rightarrow \mu
\end{array}
$$

In much of the theory (including BCS), $U(1)_{EM}$ is treated as a global symmetry for purposes of calculations of gap, condensate, etc.; it later can be weakly gauged.

In this spirit, consider a field theory on $\mathbb{R}^{2,1}$
with a global $U(1)$ and an $AdS_4$ dual:

$$L = R + \frac{6}{L^2} - \frac{1}{4} F_{\mu\nu}^2 + \left( \text{matter fields both charged and uncharged} \right)$$

$F_{\mu\nu}$ represents bulk gauging of the global $U(1)$ on the boundary.

S-wave case: matter fields include

$$L_\varphi = -\frac{1}{(\sigma-i\varphi)}|\varphi|^2 - m^2 |\varphi|^2 + \ldots$$

In the normal state, $\varphi = 0$ because only $\varphi = 0$ is preserved by $U(1)$ rotations. Simplest solution is then $RNA_{AdS_4}$ (again see lectures by H. Liu).
\[ E_z = F_{oz} = -\partial_z \Phi \]

where \( A_0 = \Phi \).

Chemical potential \( \mu = \Phi_{\text{body}} - \Phi_{\text{horizon}} \)

usually chosen to be 0 so that
\( A = \Phi dt \) is well-defined @ horizon

Will the scalar condense outside the horizon?

Heuristically: Yes, provided \( q \neq 0 \), \( m \) is not too big, and \( T \) is small enough.

If \( qE > mg \), then quantum of \( \Phi \) wants to go up:
Nothing can escape from AdS

\( g = 2 \pi T \)

Hawking, so we expect a condensate \( \phi \neq 0 \) (spontaneously breaking U(1)!) for \( T < T_c \) for some critical \( T_c \).

\textit{p-wave case:} Instead of using scalar \( \phi \) as charged matter field, let's promote \( F_{\mu\nu} \) to an \( SU(2) \) field strength. If original U(1)
is associated with $T^3$ part of $SU(2)$, then
\[ A^\pm_\mu = A^1_\mu \pm i A^2_\mu \] are fields with charges
\[ q = \pm g Y_m \] : this is just about how $W^\pm$
bosons arise, except here we have no Higgs
field and no $U(1)$ of hypercharge.

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Domain wall backgrounds in the probe limit

In the limit $g_{YM} \to \infty$, the gauge field doesn't
back-react on the geometry. To see this,

define \[ \hat{A}_\mu = \frac{A_\mu}{g_{YM}} \]
\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu] \]

\[ \mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4g_{YM}^2} tr \hat{F}_{\mu\nu}^2 \]
Suppression means we can solve eom's of \( L_{\text{grav}} = R + \frac{G_m}{L^2} \) to get \((@ T=0)\)

\[
\text{AdS}_4: \quad ds^2 = \frac{L^2}{z^2} \left[ -dt^2 + (dx')^2 + (dx^2)^2 + dz^2 \right]
\]

and then solve classical YM equations in this background. AdS\(_4\) is conformal to \(\mathbb{R}^{3,1}\):

\[
ds^2 = -dt^2 + (dx')^2 + (dx^2)^2 + dz^2
\]

and classical YM equations are conformally invariant. So we can solve on \(\mathbb{R}^{3,1}\).

Ansatz: \( A = \Phi \int^\lambda T^3 \, dt + W T' \, dx' \)

- Need this part to describe \(U(1)\) chemical potential
- This is the simplest term that spontaneously breaks that \(U(1)\)
Find \( \frac{d^2 \Phi}{d Z^2} = W^2 \Phi \), \( \frac{d^2 W}{d Z^2} = -\Phi^2 W \).

Demand \( \Phi \rightarrow 0, \ W \rightarrow W IR \), \( \Phi \rightarrow \mu, \ W \rightarrow 0 \) as \( Z \rightarrow 0 \) IR

\( \Phi \rightarrow \mu, \ W \rightarrow 0 \) as \( Z \rightarrow \infty \) UV

Because symmetry breaking is supposed to be spontaneous.

The solution with these boundary conditions is essentially unique.

Soon we'll want to make a further approximation: replace \( \Phi \) and \( W \) by step functions.
\[ W_{\text{mdden}}(z) = W_{IR} \Theta(z_* - z) \]
\[ \Phi_{\text{mdden}}(z) = \mu \Theta(z - z_*) \]

Exercise: Solve the YM equations and give numerical values for \( \frac{W_{IR}}{\mu} \) and \( W_{IR} z_* \), defining \( z_* \), so that

\[ \int_0^\infty d\zeta \Phi_{\text{mdden}}(\zeta) = \int_0^\infty d\zeta \Phi(\zeta) \]
Fermion correlators and the sudden approximation

The massless Dirac equation is essentially invariant under conformal transformations too:

Defining \( \Psi = \left( \frac{L}{Z} \right)^{3/2} \psi \), one finds

\[
\Gamma^\mu (\gamma_\mu - i \hat{A}_\mu) \psi = 0 \iff \Gamma^\mu (\theta_\mu - i \hat{A}_\mu) \psi = 0
\]

Ansatz: \( \Psi(x^m, z) = e^{ik_m x^m} \hat{\Psi}(z) \)

Recalling \( K_m \equiv k_m - \hat{A}_m \), we find

\[
(\Gamma^m i K_m + \Gamma^z A_z) \psi = 0 \quad \text{and} \quad \Gamma^{z} \Gamma^{z} = 1,
\]
\[
(\partial_z + i \Gamma^z \Gamma^m K_m) \hat{\Psi} = 0
\]

Formally, \( \hat{\Psi}(z) = P \left\{ e^{-\int_0^z \Gamma^z \Gamma^m K_m(z')} \right\} \hat{\Psi}(0) \)

is the general solution. The solution we want for computing Green's functions has asymptotic behavior

\[
\hat{\Psi} \propto e^{-K_{IR} z} \Psi \quad \text{for large } z,
\]

where \( \Psi \) is a constant spinor.

Exercise: Demonstrate that this is correct by showing that \( \hat{\Psi} \to 0 \) as \( z \to \infty \) when \( K_{IR} \) is Euclidean (easy) and that
using the \( +iE \) prescription appropriate for\nretarded Green's functions makes \( e^{-K_{IR}z} u \) the infalling solution.

Noting that \( (\partial_{z} + K_{IR}) e^{-K_{IR}z} u = 0 \), we \nsee that the condition on \( \hat{\psi}(z) \) is

\[
\lim_{z \to \infty} (K_{IR} - i \Gamma^{z} \Gamma^{m} K_{m}) \hat{\psi}(z) = 0
\]

which is equivalent to

\[
(K_{IR} - i \Gamma^{z} \Gamma^{m} K_{m}) \mathcal{P} \left\{ e^{-\int_{0}^{\infty}dz \Gamma^{z} \Gamma^{m} K_{m}(z)} \right\} \hat{\psi}(0)
\]

\[= 0\]

Formally, this is just \( \mathcal{P} \hat{\psi}(0) = 0 \) for a \nmatrix \( \mathcal{P} \) that has 4-valued Dirac and 2-valued
SU(2) indices.

The usual basis for $\Gamma^m$ in this type of calculation is:

\[
\Gamma^m = \begin{pmatrix} 0 & \gamma^m \\ \gamma^m & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[\gamma^t = i \sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_3, \quad \gamma^3 = 2 \gamma^{mn} = 2 \text{ diag } \{ -1, 1, 1 \}
\]

If in this basis we express

\[
P = \begin{pmatrix} P^{++} & P^+_- \\ P^-_+ & P^{--} \end{pmatrix},
\]

then

\[
G(k) = i P^{-_+} P^{++} \gamma^t = i P^{--} P^-_+ \gamma^t
\]

Exercise: Demonstrate the equalities.
Because $K_m(z)$'s don't commute at different values of $z$, we need something extra to make further progress. So let's use the sudden approximation:

$$
\Phi_{\text{sudden}}(z) = \begin{cases} 
    e^{-iz \Gamma^m K_m^{uv}} \hat{\Phi}(0) & \text{for } 0 < z < z_*^c \\
    e^{-i(z-z_*) \Gamma^m K_m^{IR}} e^{-iz_* \Gamma^m K_m^{IR}} \Phi(0) & \text{for } z > z_*^c 
\end{cases}
$$

where we take advantage of the fact that the $K_m^{IR}$ commute with one another, as do the $K_m^{uv}$.

$$(K_m^{IR} - i \Gamma^m K_m^{IR}) \Phi_{\text{sudden}}(z) = 0 \quad \text{for all } z > z_*^c$$

This to compute the sudden approximation
Gubser t2 p.21

Gudden (k) to the fermion 2pt function G(k), we can use

\[ P = (K_{IR} - i \Gamma_{IR}^{m} K_{m}) e^{-i z_{*} \Gamma_{m} K_{uv}} \]

\[ = q + \Gamma_{m} q_{m} + \Gamma_{m} \Gamma_{n} q_{mn} \]

where

\[ q = K_{IR} \cosh(z_{*} K_{uv}) \]

\[ q_{m} = -i \left[ K_{m} \cosh(z_{*} K_{uv}) + K_{IR} K_{uv} \frac{\sinh(z_{*} K_{uv})}{K_{uv}} \right] \]

\[ q_{mn} = K_{m} K_{n} \frac{\sinh(z_{*} K_{uv})}{K_{uv}} \]

Now just a bit more work with \( y^{m} \)-matrices allows us to demonstrate the result I claimed earlier:

\[ \chi_{\text{Gudden}}(k) = -i (y^{m} q_{m})^{-1} (q + y^{m} y^{n} q_{mn}) y^{k} \]
\( q_i, q_m, \) and \( q_{mn} \) are analytic functions of \( K^\mu_\nu \),

because \( \cosh(z^*_\mu K^\mu_\nu) \) and \( \sinh(z^*_\mu K^\mu_\nu) \) are

\( K^\mu_\nu \)

really functions of \( K^2_\mu \equiv q_{mn} K^\mu_\nu K^\nu_\mu \).

\textbf{Spectral weight of } \mathcal{G}^\Phi \textbf{ arises from the poles and/or branch cuts of } \mathcal{G} \textbf{ as a function of the } k_m. \text{ As I already reviewed, branch cuts can only come from the square root in } K^\mu_\nu = \sqrt{q_{mn} K^\mu_\nu K^\nu_\mu}, \text{ i.e. if } K^\mu_\nu \text{ is}

timelike. In such a case, } K^\mu_\nu \text{ is imaginary and } e^{-K^\mu_\nu z^\nu} \text{ is oscillatory, infalling if we're computing } \mathcal{G}_R(k).
More explicitly: \( \hat{A}_i^\text{IR} = W \hat{T}_1 \), whose eigenvalues are \( \pm k_\star \), where \( k_\star = \frac{W^2}{2} \).

\( K^\text{IR}_m \) has eigenvalues \( k_m - k_{\lambda, m} \) where

\[
(k_{\lambda, m}^\pm, 0) = (0, \pm k_\star, 0)
\]

What really matters is if \( k_m - k_{\lambda, m} \) is timelike:

if so then there is a branch cut in the 2 eigenspace of \( K^\text{IR}_m \), and hence in \( G^\Psi \).

But there is another way to get spectral weight:

\[
(y \cdot q_m)^{-1} \text{ might have a pole!}
\]
If \( K_{IR} \) has a timelike part, then \( K_{IR} \) has an anti-hermitian part, and it would be non-generic for

\[
\gamma^m q_m = -i \left[ \gamma^m K_{IR} \cosh(z^* K_{uv}) + K_{IR} \gamma^m K_{uv} \cdot \frac{\sinh(z^* K_{uv})}{K_{uv}} \right]
\]

to be non-invertible.

But if \( K_{IR} \) is hermitian, adjusting one parameter (e.g., \( \omega \) with \( k^1 \) & \( k^2 \) fixed) will make

\[
\det \gamma^m q_m = 0.
\]

This is not an airtight argument! But it does capture the correct conclusion: there's a pole in \( G(\psi) \) outside the Dirac cones,
but when it passes inside it becomes a sharp but finite-width resonance.

As with topic 1, I've left out a lot, both in the actual computations explained and in possible extensions, related computations, and comparisons with real-world phenomena. In particular:

- What happens when you include back-reaction of the gauge field on the geometry?

- The sudden approximation is not controlled by a small parameter (except maybe in some corners of k-space). How close is it to
the true $G^\Psi(k)$?

- How does $G^\Psi(k)$ change as we go from $T = 0$ to $T = T_c$ for superconductivity?

- Little seemed to depend on the choice of gauge group. How about $SO(4)$ with a vector 4 fermion? Any relation to the $SO(4)$ symmetry of Hubbard on a bipartite lattice?

- Branch cut structure is already in strict IR limit — could have been more explicit on this point.

- Poles in $G(k)$ correspond to normal modes where $\Psi \to 0$ both for $z \to 0$ & $z \to \infty$ — could again have been more explicit on this
- Normal modes are restricted to "preferred region" where $K^\text{IR}_m$ is spacelike but $K^\text{UV}_m$ is timelike.

- There's a recent extension to a d-wave condensate, see Benini et al's 1006.0731. Also has the advantage of highly anisotropic Dirac cones.

- I've omitted discussion of the significant literature on
  - fermions in $S$-wave holographic superconductors
  - conductivity at finite frequency
  - thermodynamic and hydrodynamic properties of $p$-wave superconductors
  - embedding holographic superconductors in string theory/M-theory
Comparison of \( G^\Psi(k) \) to results of ARPES measurements is interesting: peak-dip-hump from normal mode plus continuum.

What about spin-3/2 fermions, e.g. the gravitini in actual supergravity theories?

There are instabilities of holographic superconductors beside the ones that spontaneously break SU(3), e.g. Gregory-Laflamme and runaway in moduli space. How do they compete?