# TASI lectures: Holography for strongly coupled media

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Below is only the skeleton of the lectures, containing the most important formulas.

## I. INTRODUCTION

One of the main themes of this school is the applications of holography in real world physics. I will talk about two possible places where holography may be useful

- Heavy ion collisions
- Unitarity fermions (nonrelativistic)

The physics of heavy ion collisions is covered in detail in Krishna Rajagopal's lectures. Here we will just mention a few facts. The RHIC experiment is a collider experiment with gold nuclei ($A = 197$). The nuclei are accelerated to gamma factor of around 100, so they look like flat pancakes in the lab frame, with thickness $\sim 0.1$ fm. They pass through each other during time $\sim 0.1$ fm, create a medium which thermalizes during time $\sim 1$ fm, and the thermalized medium persist until a later time, perhaps $\sim 5 - 10$ fm. The hydrodynamic theory is applicable between the latter two time scales.

We will describe the unitarity fermions at the relevant place.

## II. THERMAL FIELD THEORY

There are two main formalisms used in thermal field theory. The first formalism is the Matsubara, Euclidean formalism. The second formalism is the real-time, close time path formalism. It is used in lattice QCD, very convenient for thermodynamic and static quantities (like correlation length), but cannot directly address dynamic, real-time quantities.

In the Matsubara formalism, the theory is formulated on a Euclidean spacetime, where the time axis is compactified to an interval $0 < \tau < \beta = 1/T$.

One can turn on source on the upper and lower parts of the contour, $J_1$ and $J_2$, and derivatives of $\log Z$ with respect to $J$ gives a $2 \times 2$ matrix propagators $G_{ab}$, where $a, b = 1, 2$.

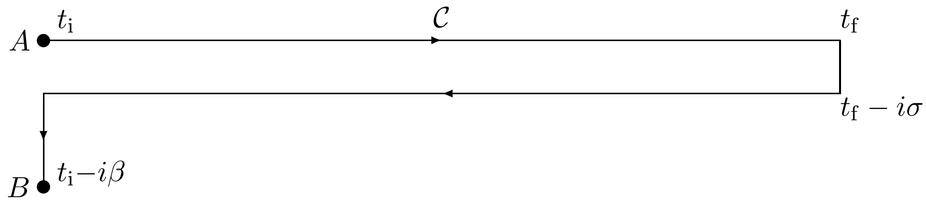


FIG. 1: The close time path contour

Changing $\sigma$ rescales the off-diagonal elements by a trivial factor,

$$G_{12}(\omega, q) = e^{\sigma\omega} G_{12}^{\sigma=0}(\omega, q), \tag{1}$$

$$G_{21}(\omega, q) = e^{-\sigma\omega} G_{21}^{\sigma=0}(\omega, q). \tag{2}$$

For $\sigma = 0$, the propagators $G_{ab}$ include path-ordered, reversed path-ordered, and Wightmann Green's functions. They are related by

$$G_{11} + G_{22} = G_{12} + G_{21} \tag{3}$$

The choice $\sigma = \beta/2$ leads to symmetric $2 \times 2$ propagator matrix: $G_{12} = G_{21}$.

The retarded propagator governs the response of a system to a small external perturbation:

$$\langle \phi(x) \rangle = -\int dy \, G_R(x - y) J(y) \tag{4}$$

## III. HYDRODYNAMICS

Hydrodynamics is a theory describing long-time, long-distance behavior of a thermal system. It is valid over time and length scales larger than the mean free time/path.

The following degrees of freedom enter the hydrodynamic description

- Conserved densities (at least $T^{00}$, $T^{0i}$, possibly densities of conserved charges)
- Goldstone bosons (superfluids)
- Unbroken U(1) gauge fields (magnetohydrodynamics)

We will consider only the case where no Goldstone or massless photon present. We also assume that there is no conserved charges, or if there are, the plasma is neutral with respect to these charges. In this case, hydrodynamics is given by the conservation equation,

$$\nabla_\mu T^{\mu\nu} = 0 \tag{5}$$

supplemented by the continuity equation that expresses $T^{\mu\nu}$ in terms of four variables: the local temperature $T$ and the local fluid velocity $u^\mu$:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu} \tag{6}$$

where $\tau^{\mu\nu}$ is the correction containing terms proportional to first derivatives. It is conventional to impose the condition $u_\mu \tau^{\mu\nu} = 0$ which eliminates any ambiguity in the definition of $u^\mu$ and $T$. In this case one has

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \zeta P^{\mu\nu} (\nabla \cdot u) \tag{7}$$

where $\eta$ and $\zeta$ are the shear and bulk viscosities, respectively. In a conformal plasma, $\epsilon = 3P \sim T^4$ and $\zeta = 0$, $\eta \sim T^3$.

### A. Kubo's formula

Turning on a small metric perturbation of the type $h_{xy}(t)$, and then measure the response of the fluid, one can derive the Kubo formula

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im}\, G_R^{xy,xy}(\omega, \mathbf{0}) \tag{8}$$

which relates the shear viscosity with a Green's function.

## IV. HOLOGRAPHIC PRESCRIPTION FOR REAL-TIME THERMAL GREEN'S FUNCTIONS

### A. Euclidean Green's function

Let us remind ourselves how the Euclidean Green's function is computed. For simplicity we limit ourselves to the case of an operator of dimension 4, dual to a massless scalar field $\phi$. Assuming the action for the scalar field is

$$S = -\frac{K}{2} \int d^5x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{9}$$

Then the prescription tells us to solve the wave equation

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{10}$$

with boundary condition $\phi = \phi_0$ at the boundary. The solution, in momentum space, is $\phi(z,k) = \phi_0(k)f_k(z)$, where $f_k(z)$ is the solution to the field equation (at momentum $k$). We now rewrite $S$ as a boundary action

$$S = \frac{K}{2}\int d^5x \frac{1}{z^3}\phi\phi' \tag{11}$$

Differentiating the action with respect to the boundary value $\phi_0$, we find the two point function to be

$$\langle\phi\phi\rangle_k \sim K \lim_{z\to 0}\frac{f_k'}{z^3} \tag{12}$$

The boundary condition at the boundary needs to be supplemented by the boundary condition in the IR. At zero temperature, we require $\phi(z)$ to vanish as $z \to 0$. At finite temperature, spacetime is capped off at some $z = z_0$. We require the field to be regular at the horizon; in the case of the scalar, $\phi'(z_0) = 0$. The solution to the

In the Poincare metric the AdS black hole looks like

$$ds^2 = -\frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f}dr^2 \tag{13}$$

where $f = 1 - r_0^4/r^4$. The metric can be extended pass the horizon, one recovered four quadrants in the following Penrose diagram,

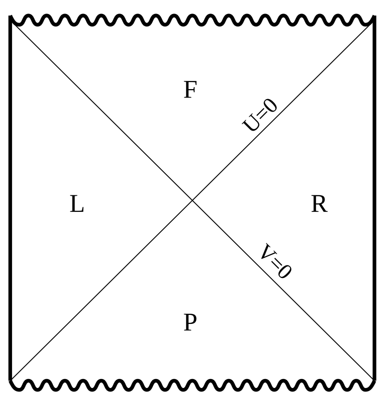


FIG. 2: Penrose diagram of AdS black hole

Let us remind ourselves how it is done. Near the horizon, we expand $r = r_0 + \rho$. The $(t, r)$ part of the metric can be rewritten as

$$ds^2 = 4\pi T\rho\left(-dt^2 + \frac{1}{(4\pi T)^2}\frac{d\rho^2}{\rho^2}\right) \tag{14}$$

where $T = r_0/\pi R^2$. This can be rewritten as $ds^2 = e^{4\pi T r_*}(-dt^2 + dr_*^2)$ where $r_* = (4\pi T)^{-1}\ln\rho$. Finally, we introduce Kruskal's coordinates

$$U = -e^{-2\pi T(t-r_*)}, \tag{15}$$

$$V = e^{2\pi T(t+r_*)} \tag{16}$$

and metric is $ds^2 = -dUdV$. The Poincare coordinates cover only $U < 0$, $V > 0$ part quadrant of the diagram. There is another copy with the same metric, corresponding to the $U > 0$, $V < 0$ part. There are two boundaries.

The extension of the AdS/CFT duality was suggested by Maldacena. The idea is that the two boundaries correspond to two horizontal parts of the close time path contour.